山东大学

硕士学位论文

正倒向重随机微分方程

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申请学位级别：硕士

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20040410

正倒向重随机微分方程

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中文摘要

1990 年， Pardoux 和 Peng[PP1] 引入了如下一般的倒向随机微分方程：

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)ds + \int_t^T Z_s dB_s, \qquad t \in [0, T].$$

从那时起，许多专家致力于这一领域的研究，倒向随机微分方程和正倒向随机微分方程得到了广泛的研究.

最近， Peng 和 Shi[PS2] 引入了一类时间对称的正倒向随机微分方程：

$$\begin{cases} y_t = x + \int_0^t f(s, y_s, Y_s, z_s, Z_s)\, ds + \int_0^t g(s, y_s, Y_s, z_s, Z_s)\, dW_s - \int_0^t z_s dB_s, \\ Y_t = \Phi(y_T) + \int_t^T F(s, y_s, Y_s, z_s, Z_s)\, ds + \int_t^T G(s, y_s, Y_s, z_s, Z_s)\, dB_s + \int_t^T Z_s dW_s. \end{cases}$$

本文中我们将研究一类被称为“正倒向重随机微分方程”的方程. 我们将减弱单调性假设，且处理更一般的情形，比如，正向方程和倒向方程的维数不同且初始条件也是函数. 我们考虑如下形式的正倒向重随机微分方程：

$$\begin{cases} y_t = \Psi(Y_0) + \int_0^t f(s, y_s, Y_s, z_s, Z_s)\, ds + \int_0^t g(s, y_s, Y_s, z_s, Z_s)\, dW_s - \int_0^t z_s dB_s, \\ Y_t = \Phi(y_T) + \int_t^T F(s, y_s, Y_s, z_s, Z_s)\, ds + \int_t^T G(s, y_s, Y_s, z_s, Z_s)\, dB_s + \int_t^T Z_s dW_s. \end{cases} \tag{0.1}$$

本文的目的是研究方程 (0.1) 解的存在唯一性以及解依赖于参数的连续性和可微性. 我们首先在单调性假设（见 (H1) 和 (H2)) 下，我们应用首先由 Peng[P1] 引入的连续性方法来处理上述情况. 另外我们在假设 (A1)-(A4) 下利用等价范数技巧和压缩映射定理得到了方程 (0.1) 解的存在唯一性.

本文共分为三部分：

第一部分：介绍本文所要研究的问题的背景.

第二部分：在假设 (H1)-(H4) 下，对于如下形式的正倒向重随机微分方程：

$$\begin{cases} dy_t = f(t, y_t, Y_t, z_t, Z_t)\,dt + g(t, y_t, Y_t, z_t, Z_t)\,dW_t - z_t dB_t, & y_0 = \Psi(Y_0), \\ dY_t = F(t, y_t, Y_t, z_t, Z_t)\,dt + G(t, y_t, Y_t, z_t, Z_t)\,dB_t + Z_t dW_t, & Y_T = \Phi(y_T). \end{cases} \tag{0.2}$$

我们通过连续性方法得到了 (0.2) 的存在唯一性定理：

定理 2.4 在假设 (H1)-(H4) 下，(0.2) 在 $M^2\left(0, T; R^{n+m+n\times l+m\times d}\right)$ 有唯一解.

第三部分：在假设 (H1)-(H5) 下，对于如下一族正倒向重随机微分方程：

$$\begin{cases} dy_t^\alpha = f_\alpha(t, y_t^\alpha, Y_t^\alpha, z_t^\alpha, Z_t^\alpha)\,dt + g_\alpha(t, y_t^\alpha, Y_t^\alpha, z_t^\alpha, Z_t^\alpha)\,dW_t - z_t^\alpha dB_t, & y_0^\alpha = \Psi_\alpha(Y_0^\alpha), \\ dY_t^\alpha = F_\alpha(t, y_t^\alpha, Y_t^\alpha, z_t^\alpha, Z_t^\alpha)\,dt + G_\alpha(t, y_t^\alpha, Y_t^\alpha, z_t^\alpha, Z_t^\alpha)\,dB_t + Z_t^\alpha dW_t, & Y_T^\alpha = \Phi_\alpha(y_T^\alpha). \end{cases} \tag{0.3}$$

我们得到了 (0.3) 的解关于参数 α 连续的结果：

定理 3.1 令 $\{f_\alpha, g_\alpha, F_\alpha, G_\alpha, \Psi_\alpha, \Phi_\alpha; \alpha \in R\}$ 为一族正倒向重随机微分方程 (0.3)，满足 (H1)-(H5), 其解记为 $(y^\alpha, Y^\alpha, z^\alpha, Z^\alpha)$. 则函数族 $\{(y^\alpha, Y^\alpha, z^\alpha, Z^\alpha); \alpha \in R\}$ 在 $M^2\left(0, T; R^{n+m+n\times l+m\times d}\right)$ 中关于 α 是连续的.

在假设 (H1)-(H4) 和 (H6) 下，对于正倒向重随机微分方程族 (0.3), 我们得到了 (0.3) 的解关于参数 α 可微性. 我们有

定理 3.2 令 $\{f_\alpha, g_\alpha, F_\alpha, G_\alpha, \Psi_\alpha, \Phi_\alpha; \alpha \in R\}$ 为一族正倒向重随机微分方程 (0.3)，且满足 (H1)-(H4) 和 (H6), 其解记为 $(y^\alpha, Y^\alpha, z^\alpha, Z^\alpha)$. 那么函数族 $\{(y^\alpha, Y^\alpha, z^\alpha, Z^\alpha); \alpha \in R\}$ 在 $M^2\left(0, T; R^{n+m+n\times l+m\times d}\right)$ 中关于 α 是可微的，且在 $\alpha = \alpha_0$ 处的导数是如下线性正倒

向重随机微分方程的解 $(\delta_\alpha y^{\alpha_0}, \delta_\alpha Y^{\alpha_0}, \delta_\alpha z^{\alpha_0}, \delta_\alpha Z^{\alpha_0})$

$$
\begin{cases}
d\delta_\alpha y_t^{\alpha_0} = [\delta_\alpha f_{\alpha_0}(t,U_t^{\alpha_0}) + \delta_y f_{\alpha_0}(t,U_t^{\alpha_0})(\delta_\alpha y_t^{\alpha_0}) + \delta_Y f_{\alpha_0}(t,U_t^{\alpha_0})(\delta_\alpha Y_t^{\alpha_0}) \\
\qquad +\delta_z f_{\alpha_0}(t,U_t^{\alpha_0})(\delta_\alpha z_t^{\alpha_0}) + \delta_Z f_{\alpha_0}(t,U_t^{\alpha_0})(\delta_\alpha Z_t^{\alpha_0})]dt \\
\qquad +[\delta_\alpha g_{\alpha_0}(t,U_t^{\alpha_0}) + \delta_y g_{\alpha_0}(t,U_t^{\alpha_0})(\delta_\alpha y_t^{\alpha_0}) + \delta_Y g_{\alpha_0}(t,U_t^{\alpha_0})(\delta_\alpha Y_t^{\alpha_0}) \\
\qquad +\delta_z g_{\alpha_0}(t,U_t^{\alpha_0})(\delta_\alpha z_t^{\alpha_0}) + \delta_Z g_{\alpha_0}(t,U_t^{\alpha_0})(\delta_\alpha Z_t^{\alpha_0})]dW_t - (\delta_\alpha z_t^{\alpha_0})dB_t, \\
d\delta_\alpha Y_t^{\alpha_0} = [\delta_\alpha F_{\alpha_0}(t,U_t^{\alpha_0}) + \delta_y F_{\alpha_0}(t,U_t^{\alpha_0})(\delta_\alpha y_t^{\alpha_0}) + \delta_Y F_{\alpha_0}(t,U_t^{\alpha_0})(\delta_\alpha Y_t^{\alpha_0}) \\
\qquad +\delta_z F_{\alpha_0}(t,U_t^{\alpha_0})(\delta_\alpha z_t^{\alpha_0}) + \delta_Z F_{\alpha_0}(t,U_t^{\alpha_0})(\delta_\alpha Z_t^{\alpha_0})]dt \\
\qquad +[\delta_\alpha G_{\alpha_0}(t,U_t^{\alpha_0}) + \delta_y G_{\alpha_0}(t,U_t^{\alpha_0})(\delta_\alpha y_t^{\alpha_0}) + \delta_Y G_{\alpha_0}(t,U_t^{\alpha_0})(\delta_\alpha Y_t^{\alpha_0}) \\
\qquad +\delta_z G_{\alpha_0}(t,U_t^{\alpha_0})(\delta_\alpha z_t^{\alpha_0}) + \delta_Z G_{\alpha_0}(t,U_t^{\alpha_0})(\delta_\alpha Z_t^{\alpha_0})]dB_t + (\delta_\alpha Z_t^{\alpha_0})dW_t, \\
\delta_\alpha y_0^{\alpha_0} = \delta_Y \Psi_{\alpha_0}(Y_0^{\alpha_0})(\delta_\alpha Y_0^{\alpha_0}) + \delta_\alpha \Psi_{\alpha_0}(Y_0^{\alpha_0}), \\
\delta_\alpha Y_T^{\alpha_0} = \delta_y \Phi_{\alpha_0}(y_T^{\alpha_0})(\delta_\alpha y_T^{\alpha_0}) + \delta_\alpha \Phi_{\alpha_0}(y_T^{\alpha_0}).
\end{cases}
\tag{0.4}
$$

第四部分：我们在假设 (A1)-(A4) 下利用等价范数技巧和压缩映射定理得到了 (0.2) 解的存在唯一性结果：

定理 4.6 假设 (A1)-(A4) 成立. 则存在 $\varepsilon_0 > 0$，它依赖于 $k_4, k_5, k_6, k_7, k_{10}, k_{11}, k_{12}, k_{14}$, λ_1, λ_2, T, 满足当 $k_1, k_3, k_8, k_9, k_{13} \in [0, \varepsilon_0)$，正倒向重随机微分方程 (0.2) 存在唯一的适应解 (y, Y, z, Z)． 进一步，如果 $\lambda_1 + \lambda_2 < \frac{-(k_7^2+k_{11}^2)}{2}$，则存在 $\varepsilon_1 > 0$，它依赖于 $k_4, k_5, k_6, k_7, k_{10}, k_{11}, k_{12}, k_{14}, \lambda_1, \lambda_2$ 且独立于 T, 满足当 $k_1, k_3, k_8, k_9, k_{13} \in [0, \varepsilon_1)$，正倒向重随机微分方程 (0.2) 存在唯一的适应解.

关键词： 随机分析，正倒向重随机微分方程，连续性方法， H- 单调，等价范数和压缩映射原理.

Forward-backward doubly stochastic differential equations

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Abstract

In 1990, Pardoux and Peng [PP1] introduced the following backward stochastic differential equation:

$$Y_t = \xi + \int_t^T g(s, Y_s, Z_s)ds + \int_t^T Z_s dB_s, \qquad t \in [0, T].$$

From then on, a lot of researchers have applied themselves to this field, backward stochastic differential equation and forward-backward stochastic differential equations have been deeply investigated.

Recently Peng and Shi [PS2] has introduced a type of time-symmetric forward-backward stochastic differential equations:

$$\begin{cases} y_t = x + \int_0^t f(s, y_s, Y_s, z_s, Z_s)\, ds + \int_0^t g(s, y_s, Y_s, z_s, Z_s)\, dW_s - \int_0^t z_s dB_s, \\ Y_t = \Phi(y_T) + \int_t^T F(s, y_s, Y_s, z_s, Z_s)\, ds + \int_t^T G(s, y_s, Y_s, z_s, Z_s)\, dB_s + \int_t^T Z_s dW_s. \end{cases}$$

In the present paper we will study this type of forward-backward doubly stochastic differential equations (FBDSDE in short),We will relax their monotonicity assumptions, and treat more general cases, for instance, the forward equations and backward equations have not same dimension and initial condition is also a function. we consider the following type of forward-backward doubly stochastic differential equations:

$$\begin{cases} y_t = \Psi(Y_0) + \int_0^t f(s, y_s, Y_s, z_s, Z_s)\, ds + \int_0^t g(s, y_s, Y_s, z_s, Z_s)\, dW_s - \int_0^t z_s dB_s, \\ Y_t = \Phi(y_T) + \int_t^T F(s, y_s, Y_s, z_s, Z_s)\, ds + \int_t^T G(s, y_s, Y_s, z_s, Z_s)\, dB_s + \int_t^T Z_s dW_s. \end{cases} \tag{0.5}$$

The aim of this paper is to study the existence and uniqueness of solution to (0.5) and the continuity and differentiability of the solutions of FBDSDE (0.5) depending on

parameters. Under some monotonicity conditions (see (H1) and (H2)), we first apply the method of continuation firstly introduced by Peng [P1], to solve (0.5). Under the conditions (A1)-(A4), we apply the technique of equivalent norm and the contraction mapping theorem to get the existence and uniqueness of solution to (0.5).

This paper is composed of four section:

In section 1, I want to introduct the papae's background.

In section 2, Under the conditions (H1)-(H4),we consider the following type of forward-backward doubly stochastic differential equations:

$$\begin{cases} dy_t = f\left(t, y_t, Y_t, z_t, Z_t\right) dt + g\left(t, y_t, Y_t, z_t, Z_t\right) dW_t - z_t dB_t, & y_0 = \Psi\left(Y_0\right), \\ dY_t = F\left(t, y_t, Y_t, z_t, Z_t\right) dt + G\left(t, y_t, Y_t, z_t, Z_t\right) dB_t + Z_t dW_t, & Y_T = \Phi\left(y_T\right). \end{cases} \tag{0.6}$$

we first apply the method of continuation to get the the existence and uniqueness of solution to (0.6).

Theorem 2.4 Under assumptions (H1)-(H4), (0.6) has a unique solution in $M^2\left(0, T; R^{n+m+n\times l+m\times d}\right)$.

In section 3,Under the conditions (H1)-(H5),we consider the following family of forward-backward doubly stochastic differential equations:

$$\begin{cases} dy_t^\alpha = f_\alpha\left(t, y_t^\alpha, Y_t^\alpha, z_t^\alpha, Z_t^\alpha\right) dt + g_\alpha\left(t, y_t^\alpha, Y_t^\alpha, z_t^\alpha, Z_t^\alpha\right) dW_t - z_t^\alpha dB_t, & y_0^\alpha = \Psi_\alpha\left(Y_0^\alpha\right), \\ dY_t^\alpha = F_\alpha\left(t, y_t^\alpha, Y_t^\alpha, z_t^\alpha, Z_t^\alpha\right) dt + G_\alpha\left(t, y_t^\alpha, Y_t^\alpha, z_t^\alpha, Z_t^\alpha\right) dB_t + Z_t^\alpha dW_t, & Y_T^\alpha = \Phi_\alpha\left(y_T^\alpha\right). \end{cases} \tag{0.7}$$

we have the following continuity of the solutions of FBDSDE depending on parameters.

Theorem 3.1 Let $\{f_\alpha, g_\alpha, F_\alpha, G_\alpha, \Psi_\alpha, \Phi_\alpha; \alpha \in R\}$ be a family of data of FBDSDE (0.7) satisfying (H1)-(H5) with the solutions denoted by $(y^\alpha, Y^\alpha, z^\alpha, Z^\alpha)$. Then the family of functions $\{(y^\alpha, Y^\alpha, z^\alpha, Z^\alpha); \alpha \in R\}$ is continuous in α in $M^2\left(0, T; R^{n+m+n\times l+m\times d}\right)$.

Under (H1)-(H4) and (H6),we discuss the differentiability of the solutions with respect to parameter. We have the following result.

Theorem 3.2 Let $\{f_\alpha, g_\alpha, F_\alpha, G_\alpha, \Psi_\alpha, \Phi_\alpha; \alpha \in R\}$ be a family of data of FBDSDE (0.7) satisfying (H1)-(H4) and (H6),with the solutions denoted by $(y^\alpha, Y^\alpha, z^\alpha, Z^\alpha)$. Then the family of functions $(y^\alpha, Y^\alpha, z^\alpha, Z^\alpha), \alpha \in R$ is differentiable in α in $M^2\left(0, T; R^{n+m+n\times l+m\times d}\right)$ and the derivatives at $\alpha = \alpha_0$ are the solution $(\delta_\alpha y^{\alpha_0}, \delta_\alpha Y^{\alpha_0}, \delta_\alpha z^{\alpha_0}, \delta_\alpha Z^{\alpha_0})$ of the following linear FBDSDE

$$\begin{cases} d\delta_\alpha y_t^{\alpha_0} = [\delta_\alpha f_{\alpha_0}(t, U_t^{\alpha_0}) + \delta_y f_{\alpha_0}(t, U_t^{\alpha_0})(\delta_\alpha y_t^{\alpha_0}) + \delta_Y f_{\alpha_0}(t, U_t^{\alpha_0})(\delta_\alpha Y_t^{\alpha_0}) \\ \qquad +\delta_z f_{\alpha_0}(t, U_t^{\alpha_0})(\delta_\alpha z_t^{\alpha_0}) + \delta_Z f_{\alpha_0}(t, U_t^{\alpha_0})(\delta_\alpha Z_t^{\alpha_0})]dt \\ \qquad +[\delta_\alpha g_{\alpha_0}(t, U_t^{\alpha_0}) + \delta_y g_{\alpha_0}(t, U_t^{\alpha_0})(\delta_\alpha y_t^{\alpha_0}) + \delta_Y g_{\alpha_0}(t, U_t^{\alpha_0})(\delta_\alpha Y_t^{\alpha_0}) \\ \qquad +\delta_z g_{\alpha_0}(t, U_t^{\alpha_0})(\delta_\alpha z_t^{\alpha_0}) + \delta_Z g_{\alpha_0}(t, U_t^{\alpha_0})(\delta_\alpha Z_t^{\alpha_0})]dW_t - (\delta_\alpha z_t^{\alpha_0})dB_t, \\ d\delta_\alpha Y_t^{\alpha_0} = [\delta_\alpha F_{\alpha_0}(t, U_t^{\alpha_0}) + \delta_y F_{\alpha_0}(t, U_t^{\alpha_0})(\delta_\alpha y_t^{\alpha_0}) + \delta_Y F_{\alpha_0}(t, U_t^{\alpha_0})(\delta_\alpha Y_t^{\alpha_0}) \\ \qquad +\delta_z F_{\alpha_0}(t, U_t^{\alpha_0})(\delta_\alpha z_t^{\alpha_0}) + \delta_Z F_{\alpha_0}(t, U_t^{\alpha_0})(\delta_\alpha Z_t^{\alpha_0})]dt \\ \qquad +[\delta_\alpha G_{\alpha_0}(t, U_t^{\alpha_0}) + \delta_y G_{\alpha_0}(t, U_t^{\alpha_0})(\delta_\alpha y_t^{\alpha_0}) + \delta_Y G_{\alpha_0}(t, U_t^{\alpha_0})(\delta_\alpha Y_t^{\alpha_0}) \\ \qquad +\delta_z G_{\alpha_0}(t, U_t^{\alpha_0})(\delta_\alpha z_t^{\alpha_0}) + \delta_Z G_{\alpha_0}(t, U_t^{\alpha_0})(\delta_\alpha Z_t^{\alpha_0})]dB_t + (\delta_\alpha Z_t^{\alpha_0})dW_t, \\ \delta_\alpha y_0^{\alpha_0} = \delta_Y \Psi_{\alpha_0}(Y_0^{\alpha_0})(\delta_\alpha Y_0^{\alpha_0}) + \delta_\alpha \Psi_{\alpha_0}(Y_0^{\alpha_0}), \\ \delta_\alpha Y_T^{\alpha_0} = \delta_y \Phi_{\alpha_0}(y_T^{\alpha_0})(\delta_\alpha y_T^{\alpha_0}) + \delta_\alpha \Phi_{\alpha_0}(y_T^{\alpha_0}). \end{cases} \tag{0.8}$$

In section 4,Under (A1)-(A4),we apply the technique of equivalent norm and the contraction mapping theorem to get the existence and uniqueness of solution to (0.6).

Theorem 4.6 Let the conditions (A1)-(A4) be satisfied. Then there exists an $\varepsilon_0 > 0$, which depends on $k_4, k_5, k_6, k_7, k_{10}, k_{11}, k_{12}, k_{14}, \lambda_1, \lambda_2, T$,such that when $k_1, k_3, k_8, k_9, k_{13} \in [0, \varepsilon_0)$,there exists a unique adapted solution (y, Y, z, Z) to the FBDSDE (0.6).Further, if $\lambda_1 + \lambda_2 < \frac{-(k_7^2+k_{11}^2)}{2}$ there is an $\varepsilon_1 > 0$, which depends on$k_4, k_5, k_6, k_7, k_{10}, k_{11}, k_{12}, k_{14}, \lambda_1, \lambda_2$ and independent of T, such that when $k_1, k_3, k_8, k_9, k_{13} \in [0, \varepsilon_1)$, there exists a unique adpted solution to the FBDSDE (0.6).

Key Words: Stochastic calculus,Forward-backward doubly stochastic differential equations,method of continuation, H-monotone,the technique of equivalent norm and the contraction mapping theorem.

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第一章　引　言

自从 1990 年 Pardoux 和 Peng[PP1] 提出了倒向随机微分方程，倒向随机微分方程和正倒向随机微分方程得到了广泛的研究．一般地，一个正倒向随机微分方程包含一个 Itô 型正向随机微分方程和一个耦合的 Pardoux-Peng 型倒向随机微分方程． Antonelli[A],Ma,Protter,Yong[MPY] 等专家进行了一系列的研究,并应用到金融中(见 [EPQ]). 其中一个方向是由 Hu 和 Peng[HP] 首先提出，由 Peng 和 Wu[PW],Yong[Y],Peng 和 Shi[PS1] 和 Peng[P2] 作了进一步的研究， Peng[P2] 推广了由 Bismut[B] 提出的随机哈密顿系统.

最近， Peng 和 Shi[PS2] 引入了一类时间对称的正倒向随机微分方程，即正向方程是关于正向随机积分 dW_t 正向的，关于倒向随机积分 dB_t 倒向的；耦合的"倒向方程"是关于倒向随机积分 dB_t 正向的，关于正向随机积分 dW_t 倒向的．换句话说，正向方程和倒向方程都是 Pardoux 和 Peng[PP2] 引入的"倒向重随机微分方程"．这些方程包含耦合的一个正向重随机微分方程和一个倒向重随机微分方程，从而推广了正倒向随机微分方程的概念.

本文中我们将研究一类被称为"正倒向重随机微分方程"的方程．它能应用于随机偏微分方程的 Feynman-Kac 公式的研究．我们将减弱单调性假设，且处理更一般的情形，比如，正向方程和倒向方程的维数不同且初始条件也是函数．我们通过由 Peng 和 Wu[PW] 引入的 H- 矩阵方法和 [S] 中的方法来处理上述情况．另外我们用等价范数技巧和压缩映射定理也处理了上述情况.

令 $(\Omega, \mathcal{F}, P)$ 是概率空间，$[0,T]$ 是一个固定的任意大的时间区间．令 $\{W_t; 0 \le t \le T\}$ 和 $\{B_t; 0 \le t \le T\}$ 是定义于 $(\Omega, \mathcal{F}, P)$，分别取值于 R^d 和 R^l 的两个完全独立的标准布朗运动．令 $\mathcal{N}$ 是 $\mathcal{F}$ 中全部 P- 零集的集合．对于每个 $t \in [0,T]$，我们定义

$$\mathcal{F}_t \doteq \mathcal{F}_t^W \vee \mathcal{F}_{t,T}^B$$

其中 $\mathcal{F}_t^W = \mathcal{N} \vee \sigma\{W_r - W_0; 0 \le r \le t\}$，$\mathcal{F}_{t,T}^B = \mathcal{N} \vee \sigma\{B_r - B_t; t \le r \le T\}$．注意 $\{\mathcal{F}_t, t \in [0,T]\}$ 是即不增也不减的，故不能构成一个信息流．设 $M^2(0,T;R^n)$ 为全部 ($dP \times dt$ 几乎处处相等)n 维联合可测且对每个 t，$\mathcal{F}_t$- 可测的随机过程 $\{v_t; t \in [0,T]\}$ 且满足

$$\mathbf{E}\int_0^T |v_t|^2\, dt < \infty.$$

的空间．显然，$M^2(0,T;R^n)$ 是 Hilbert 空间．对于给定的 $u \in M^2(0,T;R^d)$ 和 $v \in$

$M^2(0,T;R^l)$, 定义（标准）正向 Itô 积分 $\int_0^{\cdot} u_s dW_s$ 和倒向 Itô 积分 $\int_{\cdot}^T v_s dB_s$. 它们都属于 $M^2(0,T;R)$. (详见 [PP2].)

我们考虑如下形式的正倒向重随机微分方程

$$\begin{cases} y_t = \Psi(Y_0) + \int_0^t f(s, y_s, Y_s, z_s, Z_s)\, ds + \int_0^t g(s, y_s, Y_s, z_s, Z_s)\, dW_s - \int_0^t z_s dB_s, \\ Y_t = \Phi(y_T) + \int_t^T F(s, y_s, Y_s, z_s, Z_s)\, ds + \int_t^T G(s, y_s, Y_s, z_s, Z_s)\, dB_s + \int_t^T Z_s dW_s. \end{cases} \tag{1.1}$$

当 (1.1) 不涉及倒向 Itô 积分时，即当 $G \equiv 0$ 和 f, g, F 是独立于 z, $\Psi(Y_0)$ 退化为独立于 Y_0 的常向量时，这个方程组将退化为由 Hu 和 Peng [HP] 等研究的正倒向随机微分方程. 另一方面， Pardoux 和 Peng [PP2] 提出了一类新的倒向随机微分方程，称为“倒向重随机微分方程”. 本文的目的是综合上述两个结果，研究 (1.1) 解的存在唯一性以及解依赖于参数的连续性和可微性. 在单调性假设下（见 (H1) 和 (H2)), 我们应用首先由 Peng [P1] 引入的连续性方法来解 (1.1). Yong [Y] 对这种方法进行了更详尽的讨论. 方程 (1.1) 推广了由 Peng 和 Shi [PS2] 引进的时间对称的正倒向随机微分方程. 另外，我们在假设 (A1)-(A4) 下利用等价范数技巧和压缩映射定理得到了 (1.1) 解的存在唯一性结果.

本文结构：在下一部分给出本文的主要结果：用连续性方法证明了正倒向重随机微分方程解的存在唯一性定理；在第三部分，我们研究正倒向重随机微分方程解依赖于参数的连续性和可微性；第四部分用等价范数技巧和压缩映射原理我们证明了解的存在唯一性.

第二章 正倒向重随机微分方程解存在唯一性定理

§2.1 预备知识

本节我们将介绍本文所需要的 [PP2] 中的倒向重随机微分方程方面的结果，主要是倒向重随机微分方程解的存在唯一性定理和推广的 Itô 公式.

对于每个 $n \in N$ ，令 $S^2([0,T];R^n)$ 为对每个 t ， $\mathcal{F}_t$- 可测的 n 维连续过程 $\{\varphi_t, t \in [0,T]\}$ 且满足

$$\mathbf{E} \sup_{0 \le t \le T} |\varphi_t|^2 dt < \infty$$

的空间. 令

$$\begin{aligned} f &: \Omega \times [0,T] \times R^k \times R^{k\times d} \longrightarrow R^k, \\ g &: \Omega \times [0,T] \times R^k \times R^{k\times d} \longrightarrow R^{k\times l}, \\ \xi &:\in L^2\left(\Omega, \mathcal{F}_T, P; R^k\right). \end{aligned}$$

我们假定存在常数 $c > 0$ 和 $0 < \alpha < 1$ 满足对任意 $(\omega, t) \in \Omega \times [0,T], (y_1, z_1), (y_2, z_2) \in R^k \times R^{k\times l}$,

$$\text{(G1)} \quad \begin{aligned} &|f(t,y_1,z_1) - f(t,y_2,z_2)|^2 \le c(|y_1 - y_2|^2 + |z_1 - z_2|^2), \\ &|g(t,y_1,z_1) - g(t,y_2,z_2)|^2 \le c|y_1 - y_2|^2 + \alpha|z_1 - z_2|^2. \end{aligned}$$

我们考虑如下倒向重随机微分方程

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s, Z_s)dB_s - \int_t^T Z_s dW_s.$$

定理 2.1 *在上面的条件下，尤其 (G1), 上述方程存在唯一解*

$$(Y, Z) \in S^2([0,T]; R^k) \times M^2(0,T; R^{k\times l}).$$

引理 2.2 *设 $\alpha \in S^2([0,T]; R^k), \beta \in M^2(0,T; R^k), \gamma \in M^2(0,T; R^{k\times l}), \delta \in M^2(0,T; R^{k\times d})$ 满足如下方程:*

$$\alpha_t = \alpha_0 + \int_0^t \beta_s ds + \int_0^t \gamma_s dW_s + \int_0^t \delta_s dB_s, \qquad 0 \le t \le T.$$

那么

$$|\alpha_t|^2 = |\alpha_0|^2 + 2\int_0^t (\alpha_s, \beta_s)ds + 2\int_0^t (\alpha_s, \gamma_s dBs) + 2\int_0^t (\alpha_s, \delta_s dWs) - \int_0^t |\gamma_s|^2 ds + \int_0^t |\delta_s|^2 ds,$$

$$\mathbf{E}|\alpha_t|^2 = \mathbf{E}|\alpha_0|^2 + 2\mathbf{E}\int_0^t (\alpha_s, \beta_s)ds - \mathbf{E}\int_0^t |\gamma_s|^2 ds + \mathbf{E}\int_0^t |\delta_s|^2 ds,$$

此外，对任一实值函数 $\phi \in C^2(R^k)$,

$$\varphi(\alpha_t) = \varphi(\alpha_0) + \int_0^t (\varphi(\alpha_s), \beta_s) ds + \int_0^t (\varphi(\alpha_s), \gamma_s dBs) + 2\int_0^t (\varphi'(\alpha_s), \delta_s dWs)$$
$$-\frac{1}{2}\int_0^t Tr[\varphi^{(2)}\gamma_s\gamma_s^*]ds + \frac{1}{2}\int_0^t Tr[\varphi^{(2)}\delta_s\delta_s^*]ds.$$

§2.2 问题的提出和主要结果

考虑如下形式的正倒向重随机微分方程

$$\begin{cases} dy_t = f(t,y_t,Y_t,z_t,Z_t)\,dt + g(t,y_t,Y_t,z_t,Z_t)\,dW_t - z_t dB_t, & y_0 = \Psi(Y_0), \\ dY_t = F(t,y_t,Y_t,z_t,Z_t)\,dt + G(t,y_t,Y_t,z_t,Z_t)\,dB_t + Z_t dW_t, & Y_T = \Phi(y_T). \end{cases} \tag{2.1}$$

其中

$$\begin{aligned}
F &: \Omega\times[0,T]\times R^n\times R^m\times R^{n\times l}\times R^{m\times d} \longrightarrow R^m,\\
f &: \Omega\times[0,T]\times R^n\times R^m\times R^{n\times l}\times R^{m\times d} \longrightarrow R^n,\\
G &: \Omega\times[0,T]\times R^n\times R^m\times R^{n\times l}\times R^{m\times d} \longrightarrow R^{m\times l},\\
g &: \Omega\times[0,T]\times R^n\times R^m\times R^{n\times l}\times R^{m\times d} \longrightarrow R^{n\times d},\\
\Psi &: \Omega\times R^m \longrightarrow R^n,\\
\Phi &: \Omega\times R^n \longrightarrow R^m.
\end{aligned}$$

我们给定一个 $m\times n$ 满秩矩阵 H, 引入记号

$$\zeta=(y,Y,z,Z), \qquad A(t,\zeta)=\left(H^TF,Hf,H^TG,Hg\right)(t,\zeta).$$

其中 $H^TG=\left(H^TG_1\cdots H^TG_l\right)$ 和 $Hg=(Hg_1\cdots Hg_d)$. 我们采用 R^n, R^m, $R^{m\times l}$ 和 $R^{n\times d}$ 里的通常的内积 $\langle\cdot,\cdot\rangle$ 和欧氏范数 $|\cdot|$. 文中所有等式和不等式都是在 $dt\times dP$ 意义下在 $[0,T]\times\Omega$ 上几乎必然成立的.

定义 2.3 *$\mathcal{F}_t$- 可测的随机过程四元组*

$$(y,Y,z,Z)\in M^2\left(0,T;R^{n+m+n\times l+m\times d}\right)$$

如果使得 (2.1) 成立, 被称为正倒向重随机微分方程 (2.1) 的解.

如下单调条件是我们的主要假设:

(H1)
$$\begin{aligned}\left\langle A(t,\zeta)-A\left(t,\bar\zeta\right),\zeta-\bar\zeta\right\rangle \le& -\mu_1\left(|H(y-\bar y)|^2+|H(z-\bar z)|^2\right)\\ &-\mu_2\left(\left|H^T\left(Y-\bar Y\right)\right|^2+\left|H^T\left(Z-\bar Z\right)\right|^2\right),\\ \forall\zeta=(y,Y,z,Z),\quad \bar\zeta=&\left(\bar y,\bar Y,\bar z,\bar Z\right)\in R^n\times R^m\times R^{n\times l}\times R^{m\times d},\qquad \forall t\in[0,T].\end{aligned}$$

(H2)
$$\begin{aligned}&\left\langle\Psi(Y)-\Psi\left(\bar Y\right),H^T\left(Y-\bar Y\right)\right\rangle\le-\beta_2\left|H^T\left(Y-\bar Y\right)\right|^2, && \forall Y,\bar Y\in R^m,\\ &\langle\Phi(y)-\Phi(\bar y),H(y-\bar y)\rangle\ge\beta_1|H(y-\bar y)|^2, && \forall y,\bar y\in R^n.\end{aligned}$$

其中 μ_1, μ_2, β_1 和 β_2 是给定的非负常数, 满足 $\mu_1+\mu_2>0$, $\beta_1+\beta_2>0$, $\mu_1+\beta_2>0$ 和 $\mu_2+\beta_1>0$. 而且要求当 $m>n$ 时, $\mu_1>0,\beta_1>0$; 当 $m<n$ 时, $\mu_2>0,\beta_2>0$.

我们还要假定:

(H3) 对每个 $\zeta \in R^{n+m+n\times l+m\times d}$, $A(\cdot,\zeta)$ 是定义在 $[0,T]$ 上 $\mathcal{F}_t$- 可测的向量过程，且 $A(\cdot,0) \in M^2\left(0,T;R^{n+m+n\times l+m\times d}\right)$ ；对每个 $y \in R^n$, $\Phi(y)$ 是 $\mathcal{F}_T$- 可测的随机向量，且 $\Phi(0) \in L^2(\Omega,\mathcal{F}_T,P;R^n)$; 对每个 $Y \in R^m$, $\Psi(Y)$ 是 $\mathcal{F}_0$ - 可测的随机向量，且 $\Psi(0) \in L^2(\Omega,\mathcal{F}_0,P;R^m)$.

(H4) $A(t,\zeta)$ 和 Φ,Ψ 满足利普希茨条件：存在常数 $k>0$ 和 $0<\lambda<1$ 满足 $\forall\zeta,\bar\zeta \in R^{n+m+n\times l+m\times d}$, $\forall t\in[0,T]$,

$$\left|f(t,y,Y,z,Z)-f\left(t,\bar y,\bar Y,\bar z,\bar Z\right)\right|^2 \le k\left(|y-\bar y|^2+\left|Y-\bar Y\right|^2+|z-\bar z|^2+\left|Z-\bar Z\right|^2\right),$$

$$\left|F(t,y,Y,z,Z)-F\left(t,\bar y,\bar Y,\bar z,\bar Z\right)\right|^2 \le k\left(|y-\bar y|^2+\left|Y-\bar Y\right|^2+|z-\bar z|^2+\left|Z-\bar Z\right|^2\right),$$

$$\left|g(t,y,Y,z,Z)-g\left(t,\bar y,\bar Y,\bar z,\bar Z\right)\right|^2 \le k\left(|y-\bar y|^2+\left|Y-\bar Y\right|^2+\left|Z-\bar Z\right|^2\right)+\lambda|z-\bar z|^2,$$

$$\left|G(t,y,Y,z,Z)-G\left(t,\bar y,\bar Y,\bar z,\bar Z\right)\right|^2 \le k\left(|y-\bar y|^2+\left|Y-\bar Y\right|^2+|z-\bar z|^2\right)+\lambda\left|Z-\bar Z\right|^2,$$

$$\left|\Psi(Y)-\Psi\left(\bar Y\right)\right| \le k\left|Y-\bar Y\right|, \qquad |\Phi(y)-\Phi(\bar y)| \le k|y-\bar y|.$$

我们的主要结果是下面的存在唯一性定理.

定理 2.4 *在假设 (H1)-(H4) 下，(2.1) 在* $M^2\left(0,T;R^{n+m+n\times l+m\times d}\right)$ *中有唯一解.*

证明：唯一性. 令 $U=(y,Y,z,Z)$ 和 $U'=(y',Y',z',Z')$ 是 (2.1) 的两个解. 我们设 $\widehat U = U-U' = \left(\widehat y,\widehat Y,\widehat z,\widehat Z\right) = (y-y',Y-Y',z-z',Z-Z')$. 在 $[0,T]$ 上对 $\langle H\widehat y,\widehat Y\rangle$ 应用 Itô 公式，我们有

$$\begin{aligned}&\mathbf{E}\langle H\widehat y_T,\Phi(y_T)-\Phi(y'_T)\rangle-\mathbf{E}\left\langle H(\Psi(Y_0)-\Psi(Y'_0)),\widehat Y_0\right\rangle\\ &\le -\mu_1\mathbf{E}\textstyle\int_0^T(|H(y-y')|^2+\left|H^T(z-z')\right|^2)dt\\ &-\mu_2\mathbf{E}\textstyle\int_0^T(\left|H^T(Y-Y')\right|^2+\left|H^T(Z-Z')\right|^2)dt.\end{aligned}$$

由假设 (H1) 和 (H2), 可得

$$\begin{aligned}&\mu_1\mathbf{E}\textstyle\int_0^T(|H(y-y')|^2+\left|H^T(z-z')\right|^2)dt\\ &+\mu_2\mathbf{E}\textstyle\int_0^T(\left|H^T(Y-Y')\right|^2+\left|H^T(Z-Z')\right|^2)dt\le 0.\end{aligned}$$

如果 $m>n$, $\mu_1>0$, 那么 $|H(y-y')|^2\equiv 0$, $|H(z-z')|^2\equiv 0$. 我们有 $y\equiv y'$ 和 $z\equiv z'$. 特别地，$\Phi(y_T)=\Phi(y'_T)$. 从而，由 [PP2] 中倒向重随机微分方程解的唯一性结果，可得 $Y\equiv Y'$ 和 $Z\equiv Z'$.

如果 $m<n$, $\mu_2>0$, 那么 $\left|H^T(Y-Y')\right|^2\equiv 0$, $\left|H^T(Z-Z')\right|^2\equiv 0$. 我们有 $Y\equiv Y'$ 和 $Z\equiv Z'$. 特别地，$\Psi(Y_0)=\Psi(Y'_0)$. 从而，由 [PP2] 中倒向重随机微分方程解的唯一性结果，可得 $y\equiv y'$ 和 $z\equiv z'$.

当 $m=n$ 时，类似于上面的情况，可得结果. 唯一性得证.

§2.3 先验估计

存在性的证明结合了上面的技巧和 [P1] 中首先引入的先验估计方法. 为使证明更清晰和易于理解, 分析三种情况: $m>n$, $m<n$, $m=n$.

第一种情形: 当 $m>n$ 时, $\mu_1>0, \beta_1>0$. 我们考虑一族正倒向重随机微分方程

$$\begin{cases} dy_t=[\alpha f(t,U_t)+f_0(t)]dt-z_t dB_t+[\alpha g(t,U_t)+g_0(t)]dW_t, \\ dY_t=[\alpha F(t,U_t)-(1-\alpha)\mu_1 Hy_t+F_0(t)]dt+Z_t dW_t \\ \qquad +[\alpha G(t,U_t)-(1-\alpha)\mu_1 Hz_t+G_0(t)]dB_t, \\ y_0=\alpha\Psi(Y_0)+\psi, \\ Y_T=\alpha\Phi(y_T)+(1-\alpha)Hy_T+\varphi, \end{cases} \tag{2.2}$$

这里 $U=(y,Y,z,Z)$ 和 $(F_0,f_0,G_0,g_0)\in M^2\left(0,T;R^{m+n+m\times l+n\times d}\right)$, $\psi\in L^2(\Omega,\mathcal{F}_0,P;R^n)$ 以及 $\varphi\in L^2(\Omega,\mathcal{F}_T,P;R^m)$ 是任意给定的.

当 $\alpha=1$ 时, 方程 (2.2) 解的存在意味着方程 (2.1) 解的存在. 当 $\alpha=0$ 时, 由倒向重随机微分方程解的存在唯一性, 方程 (2.2) 是唯一可解的. 下面的引理在连续性方法中是关键的, 它给出了方程 (2.2) 对 $\alpha=\alpha_0\in[0,1)$, 解存在区间的估计.

引理 2.5 *假设 $m>n$, (H1)-(H4), 若某个 $\alpha_0\in[0,1)$, $\psi\in L^2(\Omega,\mathcal{F}_0,P;R^n)$, $\varphi\in L^2(\Omega,\mathcal{F}_T,P;R^m)$, $(F_0,f_0,G_0,g_0)\in M^2\left(0,T;R^{m+n+m\times l+n\times d}\right)$, 方程 (2.2) 解存在, 则存在某一与 α_0 无关的正常数 δ_0, 使对 $\alpha\in[\alpha_0,\alpha_0+\delta_0]$, $\psi\in L^2(\Omega,\mathcal{F}_0,P;R^n)$, $\varphi\in L^2(\Omega,\mathcal{F}_T,P;R^m)$, $(F_0,f_0,G_0,g_0)\in M^2\left(0,T;R^{m+n+m\times l+n\times d}\right)$, 方程 (2.2) 也唯一可解.*

证明: 由 $\psi\in L^2(\Omega,\mathcal{F}_0,P;R^n)$, $\varphi\in L^2(\Omega,\mathcal{F}_T,P;R^m)$, (F_0,f_0,G_0,g_0) $\in M^2\left(0,T;R^{m+n+m\times l+n\times d}\right)$, $\alpha=\alpha_0$, 方程 (2.2) 存在唯一解, 则对每个 $\bar U=\left(\bar y,\bar Y,\bar z,\bar Z\right)\in M^2\left(0,T;R^{n+m+n\times l+m\times d}\right)$, 存在 $U=(y,Y,z,Z)\in M^2\left(0,T;R^{n+m+n\times l+m\times d}\right)$ 满足如下的方程:

$$\begin{aligned} dy_t &= \left[\alpha_0 f(t,U_t)+\delta f\left(t,\bar U_t\right)+f_0(t)\right]dt-z_t dB_t \\ &\quad +\left[\alpha_0 g(t,U_t)+\delta g\left(t,\bar U_t\right)+g_0(t)\right]dW_t, \\ dY_t &= \left[\alpha_0 F(t,U_t)-(1-\alpha_0)\mu_1 Hy_t+\delta\left(F\left(t,\bar U_t\right)+\mu_1 H\bar y_t\right)+F_0(t)\right]dt+Z_t dW_t \\ &\quad +\left[\alpha_0 G(t,U_t)-(1-\alpha_0)\mu_1 Hz_t+\delta\left(G\left(t,\bar U_t\right)+\mu_1 H\bar z_t\right)+G_0(t)\right]dB_t, \\ y_0 &= \alpha_0\Psi(Y_0)+\delta\Psi(\bar Y_0)+\psi, \\ Y_T &= \alpha_0\Phi(y_T)+(1-\alpha_0)Hy_T+\delta\left(\Phi(\bar y_T)-H\bar y_T\right)+\varphi, \end{aligned}$$

这里 δ 是独立于 α_0 不超过 1 的正常数.

我们的目的是证明下面的映射:

$$U = I_{\alpha_0+\delta}(\bar{U}) : M^2\left(0,T;R^{n+m+n\times l+m\times d}\right) \to M^2\left(0,T;R^{n+m+n\times l+m\times d}\right)$$

对于足够小的 δ 是压缩的.

令 $\bar{U}' = (\bar{y}', \bar{Y}', \bar{z}', \bar{Z}') \in M^2\left(0,T;R^{n+m+n\times l+m\times d}\right)$,$(y', Y', z', Z') = U' = I_{\alpha_0+\delta}\left(\bar{U}'\right)$,

$$\begin{aligned}\widehat{\bar{U}} &= \bar{U} - \bar{U}' = \left(\widehat{\bar{y}}, \widehat{\bar{Y}}, \widehat{\bar{z}}, \widehat{\bar{Z}}\right) = (\bar{y} - \bar{y}', \bar{Y} - \bar{Y}', \bar{z} - \bar{z}', \bar{Z} - \bar{Z}'),\\ \widehat{U} &= U - U' = \left(\widehat{y}, \widehat{Y}, \widehat{z}, \widehat{Z}\right) = (y - y', Y - Y', z - z', Z - Z').\end{aligned}$$

在 $[0,T]$ 上对 $\langle H\widehat{y}, \widehat{Y}\rangle$ 应用 Itô 公式，得到

$$\begin{aligned}&\mathbf{E}\left\langle H\widehat{y}_T, \alpha_0\widehat{\Phi}\left(y_T\right) + (1-\alpha_0)H\widehat{y}_T\right\rangle - \mathbf{E}\int_0^T \left\langle \alpha_0\left(A\left(t,U_t\right) - A\left(t,U_t'\right)\right), \hat{U}_t\right\rangle dt\\ &+(1-\alpha_0)\mu_1\mathbf{E}\int_0^T \left(|H\widehat{y}_t|^2 + |H\widehat{z}_t|^2\right) dt\\ =\ &\mathbf{E}\left\langle H\widehat{y}_T, \delta H\widehat{\bar{y}}_T\right\rangle - \mathbf{E}\left\langle H\widehat{y}_T, \delta\widehat{\Phi}(\bar{y}_T)\right\rangle + \mathbf{E}\left\langle H\left(\alpha_0\widehat{\Psi}(Y_0) + \delta\widehat{\Psi}(\bar{Y}_0)\right), \widehat{Y}_0\right\rangle\\ &+\delta\mathbf{E}\int_0^T \left(\left\langle \widehat{Y}_t, H\widehat{f}\left(t,\bar{U}_t\right)\right\rangle + \left\langle H\widehat{y}, \widehat{F}\left(t,\bar{U}_t\right)\right\rangle + \left\langle \widehat{Z}_t, H\widehat{g}\left(t,\bar{U}_t\right)\right\rangle + \left\langle H\widehat{z}_t, \widehat{G}\left(t,\bar{U}_t\right)\right\rangle\right) dt\\ &+\delta\mu_1\mathbf{E}\int_0^T \left(\langle H\widehat{y}_t, H\widehat{\bar{y}}_t\rangle + \langle H\widehat{z}_t, H\widehat{\bar{z}}_t\rangle\right) dt,\end{aligned}$$

其中

$$\begin{aligned}\widehat{f} &= f\left(t,U_t\right) - f\left(t,U_t'\right),\\ \widehat{g} &= g\left(t,U_t\right) - g\left(t,U_t'\right),\\ \widehat{F} &= F\left(t,U_t\right) - F\left(t,U_t'\right),\\ \widehat{G} &= G\left(t,U_t\right) - G\left(t,U_t'\right).\end{aligned}$$

由 $m > n$, (H1)-(H4), 我们易得

$$\begin{aligned}&(1-\alpha_0+\alpha_0\beta_1)\mathbf{E}|H\widehat{y}_T|^2 + \mu_1\mathbf{E}\int_0^T \left(|H\widehat{y}_t|^2 + |H\widehat{z}_t|^2\right) dt\\ &\leq \delta C\mathbf{E}\int_0^T \left(\left|\widehat{U}_t\right|^2 + \left|\widehat{\bar{U}}_t\right|^2\right) dt + \delta C\left(\mathbf{E}|\hat{Y}_0|^2 + \mathbf{E}|\widehat{y}_T|^2 + \mathbf{E}|\widehat{\bar{Y}}_0|^2 + \mathbf{E}|\widehat{\bar{y}}_T|^2\right)\end{aligned} \tag{2.3}$$

这里 $C > 0$ 的常数. 此后， C 将是适当的常数，它可以逐行不同且只依赖于利普希茨常数 k, H, μ_1, β_1, λ 及 T.

另外，对 $\left(\widehat{Y},\widehat{Z}\right)=(Y-Y',Z-Z')$ 应用估计的通常技术. 在 $[t,T]$ 上对 $\left|\widehat{Y}_t\right|^2$ 应用 Itô 公式，得到

$$
\begin{aligned}
&\mathbf{E}\left|\widehat{Y}_T\right|^2-\mathbf{E}\left|\widehat{Y}_t\right|^2\\
=\;&2\mathbf{E}\int_t^T\left\langle\widehat{Y}_s,\alpha_0\widehat{F}\left(s,U_s\right)-(1-\alpha_0)\,\mu_1H\widehat{y}_s+\delta\left(\widehat{F}\left(s,\bar{U}_s\right)+\mu_1H\widehat{\bar{y}}_s\right)\right\rangle ds\\
&-\mathbf{E}\int_t^T\left|\alpha_0\widehat{G}\left(s,U_s\right)-(1-\alpha_0)\mu_1H\widehat{z}_s+\delta\left(\widehat{G}\left(s,\bar{U}_s\right)+\mu_1H\widehat{\bar{z}}_s\right)\right|^2ds+\mathbf{E}\int_t^T\left|\widehat{Z}_s\right|^2ds.
\end{aligned}
$$

由 (H4) 我们有

$$
\begin{aligned}
&\mathbf{E}\left|\widehat{Y}_t\right|^2+\mathbf{E}\int_t^T\left|\hat{Z}_s\right|^2ds\\
=\;&\mathbf{E}\left|\alpha_0\widehat{\Phi}\left(y_T\right)+(1-\alpha_0)H\widehat{y}_T+\delta\left(\widehat{\Phi}\left(\bar{y}_T\right)-H\widehat{\bar{y}}_T\right)\right|^2\\
&-2\mathbf{E}\int_t^T\left\langle\widehat{Y}_s,\alpha_0\widehat{F}\left(s,U_s\right)-(1-\alpha_0)\,\mu_1H\widehat{y}_s+\delta\left(\widehat{F}\left(s,\bar{U}_s\right)+\mu_1H\widehat{\bar{y}}_s\right)\right\rangle ds\\
&+\mathbf{E}\int_t^T\left|\alpha_0\widehat{G}\left(s,U_s\right)-(1-\alpha_0)\mu_1H\widehat{z}_s+\delta\left(\widehat{G}\left(s,\bar{U}_s\right)+\mu_1H\widehat{\bar{z}}_s\right)\right|^2ds\\
\leq\;&4\mathbf{E}\left[\alpha_0^2\left|\widehat{\Phi}\left(y_T\right)\right|^2+(1-\alpha_0)^2\left|H\widehat{y}_T\right|^2+\delta^2\left|\widehat{\Phi}\left(\bar{y}_T\right)\right|^2+\delta^2\left|H\widehat{\bar{y}}_T\right|^2\right]\\
&+2\mathbf{E}\int_t^T\left|\widehat{Y}_s\right|\left(\alpha_0\left|\widehat{F}\left(s,U_s\right)\right|+(1-\alpha_0)\,\mu_1\left|H\widehat{y}_s\right|+\delta\left|\widehat{F}\left(s,\bar{U}_s\right)\right|+\delta\mu_1\left|H\widehat{\bar{y}}_s\right|\right)ds\\
&+\mathbf{E}\int_t^T\left[\frac{1+\lambda}{2\lambda}\alpha_0^2\left|\widehat{G}\left(s,U_s\right)\right|^2\right]ds\\
&+\mathbf{E}\int_t^T\left[\frac{1+\lambda}{1-\lambda}\left((1-\alpha_0)^2\mu_1^2\left|H\widehat{z}_s\right|^2+\delta^2\left|\widehat{G}\left(s,\bar{U}_s\right)\right|^2+\delta^2\mu_1^2\left|H\widehat{\bar{z}}_s\right|^2\right)\right]ds\\
\leq\;&C\mathbf{E}\left|\widehat{y}_T\right|^2+\delta C\mathbf{E}\left|\widehat{\bar{y}}_T\right|^2\\
&+\mathbf{E}\int_t^T\left[\left(\frac{4k}{1-\lambda}\left|\widehat{Y}_s\right|^2+\frac{1-\lambda}{4k}\left|\widehat{F}\left(s,U_s\right)\right|^2\right)+(1-\alpha_0)\,\mu_1\left(\left|\widehat{Y}_s\right|^2+\left|H\widehat{y}_s\right|^2\right)\right]ds\\
&+\delta\mathbf{E}\int_t^T\left[\left(\left|\widehat{Y}_s\right|^2+\left|\widehat{F}\left(s,\bar{U}_s\right)\right|^2\right)+\mu_1\left(\left|\widehat{Y}_s\right|^2+\left|H\widehat{\bar{y}}_s\right|^2\right)\right]ds\\
&+\mathbf{E}\int_t^T\left[\frac{1+\lambda}{2\lambda}\left|\widehat{G}\left(s,U_s\right)\right|^2\right]ds\\
&+\mathbf{E}\int_t^T\left[\frac{1+\lambda}{1-\lambda}\left((1-\alpha_0)^2\mu_1^2\left|H\widehat{z}_s\right|^2+\delta^2\left|\widehat{G}\left(s,\bar{U}_s\right)\right|^2+\delta^2\mu_1^2\left|H\widehat{\bar{z}}_s\right|^2\right)\right]ds\\
\leq\;&C\mathbf{E}\left|\widehat{y}_T\right|^2+\delta C\mathbf{E}\left|\widehat{\bar{y}}_T\right|^2+C\mathbf{E}\int_t^T\left|\widehat{Y}_s\right|^2ds+\delta C\mathbf{E}\int_t^T\left|\widehat{\bar{U}}_s\right|^2ds\\
&+C\mathbf{E}\int_t^T\left(\left|\widehat{y}_s\right|^2+\left|\widehat{z}_s\right|^2\right)ds+\frac{3+\lambda}{4}\mathbf{E}\int_t^T\left|\hat{Z}_s\right|^2ds.
\end{aligned}
$$

从而我们有

$$\begin{aligned}&\mathbf{E}\left|\widehat{Y}_t\right|^2+\frac{1-\lambda}{4}\mathbf{E}\int_t^T\left|\hat{Z}_s\right|^2ds\\ \leq\ & C\mathbf{E}\int_t^T\left|\widehat{Y}_s\right|^2ds+C\left(\mathbf{E}\left|\widehat{y}_T\right|^2+\delta\mathbf{E}\left|\widehat{\overline{y}}_T\right|^2\right)+C\mathbf{E}\int_t^T\left(\left|\widehat{y}_s\right|^2+\left|\widehat{z}_s\right|^2+\delta\left|\widehat{\overline{U}}_s\right|^2\right)ds.\end{aligned}$$

由 Gronwall 不等式，可得

$$\begin{aligned}&\mathbf{E}\left|\widehat{Y}_t\right|^2+\mathbf{E}\int_t^T\left|\hat{Z}_s\right|^2ds\\ \leq\ & C\left(\mathbf{E}\left|\widehat{y}_T\right|^2+\delta\mathbf{E}\left|\widehat{\overline{y}}_T\right|^2\right)+C\mathbf{E}\int_0^T\left(\left|\widehat{y}_t\right|^2+\left|\widehat{z}_t\right|^2+\delta\left|\widehat{\overline{U}}_t\right|^2\right)dt,\quad\forall t\in[0,T].\end{aligned}$$

则我们得到

$$\begin{aligned}&\mathbf{E}\left|\widehat{Y}_0\right|^2+\mathbf{E}\textstyle\int_0^T\left(\left|\widehat{Y}_t\right|^2+\left|\hat{Z}_t\right|^2\right)dt\\ &\leq C\left(\mathbf{E}\left|\widehat{y}_T\right|^2+\delta\mathbf{E}\left|\widehat{\overline{y}}_T\right|^2\right)+C\mathbf{E}\textstyle\int_0^T\left(\left|\widehat{y}_t\right|^2+\left|\widehat{z}_t\right|^2+\delta\left|\widehat{\overline{U}}_t\right|^2\right)dt.\end{aligned}\tag{2.4}$$

结合上面的两个估计 (2.3) 和 (2.4), 对于足够大的常数 C 我们易得

$$\begin{aligned}&\mathbf{E}\int_0^T\left|\widehat{U}_t\right|^2dt+\mathbf{E}\left|\widehat{y}_T\right|^2+\mathbf{E}\left|\widehat{Y}_0\right|^2\\ \leq\ & \delta C\left(\mathbf{E}\int_0^T\left|\widehat{U}_t\right|^2dt+\mathbf{E}\left|\widehat{y}_T\right|^2+\mathbf{E}\left|\widehat{Y}_0\right|^2+\mathbf{E}\int_0^T\left|\widehat{\overline{U}}_t\right|^2dt+\mathbf{E}\left|\widehat{\overline{y}}_T\right|^2+\mathbf{E}\left|\widehat{\overline{Y}}_0\right|^2\right).\end{aligned}$$

我们取 $\delta_0=\frac{1}{3C}$. 容易看出，对于固定的 $\delta\in[0,\delta_0]$, 映射 $I_{\alpha_0+\delta}$ 是压缩的

$$\mathbf{E}\int_0^T\left|\widehat{U}_t\right|^2dt+\mathbf{E}\left|\widehat{y}_T\right|^2+\mathbf{E}\left|\widehat{Y}_0\right|^2\leq\frac{1}{2}\left(\mathbf{E}\int_0^T\left|\widehat{\overline{U}}_t\right|^2dt+\mathbf{E}\left|\widehat{\overline{y}}_T\right|^2+\mathbf{E}\left|\widehat{\overline{Y}}_0\right|^2\right).$$

从而这个映射存在唯一不动点 $U=(y,Y,z,Z)\in M^2\left(0,T;R^{n+m+n\times l+m\times d}\right)$, 它就是当 $\alpha=\alpha_0+\delta$, $\delta\in[0,\delta_0]$, 方程 (2.2) 的解. 引理 3 得证.

第二种情形：当 $m<n$ 时， $\mu_2>0$, $\beta_2>0$. 我们考虑一族正倒向重随机微分方程

$$\begin{cases}dy_t=\left[\alpha f(t,U_t)-(1-\alpha)\mu_2H^TY_t+f_0(t)\right]dt-z_tdB_t\\ \qquad+\left[\alpha g(t,U_t)-(1-\alpha)\mu_2H^TZ_t+g_0(t)\right]dW_t,\\ dY_t=\left[\alpha F(t,U_t)+F_0(t)\right]dt+Z_tdW_t+\left[\alpha G(t,U_t)+G_0(t)\right]dB_t,\\ y_0=\alpha\Psi(Y_0)+(1-\alpha)H^TY_0+\psi,\\ Y_T=\alpha\Phi(y_T)+\varphi.\end{cases}\tag{2.5}$$

这里 $U=(y,Y,z,Z)$ 和 $(F_0,f_0,G_0,g_0)\in M^2\left(0,T;R^{m+n+m\times l+n\times d}\right)$, $\psi\in L^2\left(\Omega,\mathcal{F}_0,P;R^n\right)$ 以及 $\varphi\in L^2\left(\Omega,\mathcal{F}_T,P;R^m\right)$ 是任意给定的.

当 $\alpha=1$ 时，方程 (2.5) 解的存在意味着方程 (2.1) 解的存在. 当 $\alpha=0$ 时，由倒向重随机微分方程解的存在唯一性，方程 (2.5) 是唯一可解的. 下面的引理在连续性方法中是关键的，它给出了方程 (2.5) 对 $\alpha=\alpha_0\in[0,1)$, 解存在区间的估计.

引理 2.6 *假设 $m<n$, (H1)-(H4), 若某个 $\alpha_0\in[0,1)$ ， $\psi\in L^2\left(\Omega,\mathcal{F}_0,P;R^n\right)$, $\varphi\in L^2\left(\Omega,\mathcal{F}_T,P;R^m\right)$, $(F_0,f_0,G_0,g_0)\in M^2\left(0,T;R^{m+n+m\times l+n\times d}\right)$ ，方程 (2.5) 解存在，则存在某一与 α_0 无关的正常数 δ_0, 使对 $\alpha\in[\alpha_0,\alpha_0+\delta_0]$, 和 $\psi\in L^2\left(\Omega,\mathcal{F}_0,P;R^n\right)$, $\varphi\in L^2\left(\Omega,\mathcal{F}_T,P;R^m\right)$, $(F_0,f_0,G_0,g_0)\in M^2\left(0,T;R^{m+n+m\times l+n\times d}\right)$, 方程 (2.5) 也唯一可解.*

证明: 由 $\psi\in L^2\left(\Omega,\mathcal{F}_0,P;R^n\right)$, $\varphi\in L^2\left(\Omega,\mathcal{F}_T,P;R^m\right)$, (F_0,f_0,G_0,g_0) $\in M^2\left(0,T;R^{m+n+m\times l+n\times d}\right)$, $\alpha=\alpha_0$, 方程 (2.5) 存在唯一解, 则对每个 $\bar{U}=\left(\bar{y},\bar{Y},\bar{z},\bar{Z}\right)\in M^2\left(0,T;R^{n+m+n\times l+m\times d}\right)$, 存在 $U=(y,Y,z,Z)\in M^2\left(0,T;R^{n+m+n\times l+m\times d}\right)$ 满足如下的方程:

$$
\begin{aligned}
dy_t &= \left[\alpha_0 f\left(t,U_t\right)-\left(1-\alpha_0\right)\mu_2 H^T Y_t+\delta\left(f\left(t,\bar{U}_t\right)+\mu_2 H^T\bar{Y}_t\right)+f_0(t)\right]dt-z_t dB_t\\
&\quad+\left[\alpha_0 g\left(t,U_t\right)-(1-\alpha_0)\mu_2 H^T Z_t+\delta\left(g\left(t,\bar{U}_t\right)+\mu_2 H^T\bar{Z}_t\right)+g_0(t)\right]dW_t,\\
dY_t &= \left[\alpha_0 F\left(t,U_t\right)+\delta F\left(t,\bar{U}_t\right)+F_0(t)\right]dt+Z_t dW_t\\
&\quad+\left[\alpha_0 G\left(t,U_t\right)+\delta G\left(t,\bar{U}_t\right)+G_0(t)\right]dB_t,\\
y_0 &= \alpha_0\Psi\left(Y_0\right)+\left(1-\alpha_0\right)H^T Y_0+\delta\left(\Psi\left(\bar{Y}_0\right)-H^T\bar{Y}_0\right)+\psi,\\
Y_T &= \alpha_0\Phi\left(y_T\right)+\delta\Phi\left(\bar{y}_T\right))+\varphi,
\end{aligned}
$$

这里 δ 是独立于 α_0 不超过 1 的正常数.

我们的目的是证明下面的映射:

$$
U=I_{\alpha_0+\delta}(\bar{U}):M^2\left(0,T;R^{n+m+n\times l+m\times d}\right)\rightarrow M^2\left(0,T;R^{n+m+n\times l+m\times d}\right)
$$

对于足够小的 δ 是压缩的. 以下类似于引理 2.5 地证明，从略.

第三种情形：当 $m=n$ 时，由 (H1) 和 (H2), 我们只需考虑:

(1) 如果 $\mu_1>0$, $\mu_2\geq 0$, $\beta_1>0$, $\beta_2\geq 0$. 我们有与引理 2.5 相同的结果,

(2) 如果 $\mu_1\geq 0$, $\mu_2>0$, $\beta_1\geq 0$, $\beta_2>0$, 可以得到引理 2.6 相同的结果.

§2.4 定理 2.4 存在性的证明

现在我们给出定理 2.4 存在性的证明.

证明: 存在性. 首先考虑 $m > n$ 的情形. 我们知道, 当 $\alpha = 0$ 时, $\psi \in L^2(\Omega, \mathcal{F}_0, P; R^n)$, $\varphi \in L^2(\Omega, \mathcal{F}_T, P; R^m)$, $(F_0, f_0, G_0, g_0) \in M^2(0, T; R^{m+n+m\times l+n\times d})$, 方程 (2.2) 有唯一解. 由引理 2.5 知, 存在与 α 无关的正常数 $\delta_0 = \delta_0(k, \lambda, \beta_1, \mu_1, H, T)$ 使得对 $\delta \in [0, \delta_0]$, $\psi \in L^2(\Omega, \mathcal{F}_0, P; R^n)$, $\varphi \in L^2(\Omega, \mathcal{F}_T, P; R^m)$, $(F_0, f_0, G_0, g_0) \in M^2(0, T; R^{m+n+m\times l+n\times d})$, 方程 (2.2) 当 $\alpha = \delta$ 时有唯一解. 由于 δ_0 只依赖于 $(k, \lambda, \beta_1, \mu_1, H, T)$, 我们可以重复 N 次这一过程, 使得 $1 \leq N\delta_0 < 1 + \delta_0$. 特别地, 当 $\alpha = 1$ 时, 取 $(F_0, f_0, G_0, g_0) \equiv 0$, $\varphi \equiv 0$, $\psi \equiv 0$, 方程 (2.2) 在 $M^2(0, T; R^{n+m+n\times l+m\times d})$ 中有唯一解.

再考虑 $m < n$ 的情形. 我们知道, 当 $\alpha = 0$ 时, $\psi \in L^2(\Omega, \mathcal{F}_0, P; R^n)$, $\varphi \in L^2(\Omega, \mathcal{F}_T, P; R^m)$, $(F_0, f_0, G_0, g_0) \in M^2(0, T; R^{m+n+m\times l+n\times d})$, 方程 (2.5) 有唯一解. 由引理 2.6 知, 存在与 α 无关的正常数 $\delta_0 = \delta_0(k, \lambda, \beta_2, \mu_2, H, T)$ 使得对 $\delta \in [0, \delta_0]$,$\psi \in L^2(\Omega, \mathcal{F}_0, P; R^n)$, $\varphi \in L^2(\Omega, \mathcal{F}_T, P; R^m)$, $(F_0, f_0, G_0, g_0) \in M^2(0, T; R^{m+n+m\times l+n\times d})$, 方程 (2.5) 当 $\alpha = \delta$ 时有唯一解. 由于 δ_0 只依赖于 $(k, \lambda, \beta_2, \mu_2, H, T)$, 我们可以重复 N 次这一过程, 使得 $1 \leq N\delta_0 < 1 + \delta_0$. 特别地, 当 $\alpha = 1$ 时, 取 $(F_0, f_0, G_0, g_0) \equiv 0$, $\varphi \equiv 0$, $\psi \equiv 0$, 方程 (2.5) 在 $M^2(0, T; R^{n+m+n\times l+m\times d})$ 中有唯一解.

类似于上面的情形, 当 $m = n$ 时, 也可得存在性结果. 存在性得证.

第三章　解依赖于参数的连续性和可微性

本节中,我们讨论 (2.1) 的解依赖于参数的连续性和可微性. 令 $\{f_\alpha, g_\alpha, F_\alpha, G_\alpha, \Psi_\alpha, \Phi_\alpha\}$, $\alpha \in R$ 为一族正倒向重随机微分方程

$$\begin{cases} dy_t^\alpha = f_\alpha\left(t, y_t^\alpha, Y_t^\alpha, z_t^\alpha, Z_t^\alpha\right) dt + g_\alpha\left(t, y_t^\alpha, Y_t^\alpha, z_t^\alpha, Z_t^\alpha\right) dW_t - z_t^\alpha dB_t, \quad y_0^\alpha = \Psi_\alpha\left(Y_0^\alpha\right), \\ dY_t^\alpha = F_\alpha\left(t, y_t^\alpha, Y_t^\alpha, z_t^\alpha, Z_t^\alpha\right) dt + G_\alpha\left(t, y_t^\alpha, Y_t^\alpha, z_t^\alpha, Z_t^\alpha\right) dB_t + Z_t^\alpha dW_t, \quad Y_T^\alpha = \Phi_\alpha\left(y_T^\alpha\right). \end{cases} \tag{3.1}$$

并满足 (H1)-(H4) 其解记为 $(y^\alpha, Y^\alpha, z^\alpha, Z^\alpha)$.

假定

$$\text{(H5)} \quad \begin{cases} (1)\ \{f_\alpha, g_\alpha, F_\alpha, G_\alpha, \Psi_\alpha, \Phi_\alpha; \alpha \in R\} \text{ 关于相同的常数 } k \text{ 满足等度利谱希茨条件;} \\ (2)\ \{f_\alpha, g_\alpha, F_\alpha, G_\alpha, \Psi_\alpha, \Phi_\alpha\} \text{在其各自空间范数意义下关于}\alpha\text{是连续的.} \end{cases}$$

我们有下面的连续性结果.

定理 3.1 　*令 $\{f_\alpha, g_\alpha, F_\alpha, G_\alpha, \Psi_\alpha, \Phi_\alpha; \alpha \in R\}$ 为一族正倒向重随机微分方程 (3.1) ,满足 (H1)-(H5), 其解记为 $(y^\alpha, Y^\alpha, z^\alpha, Z^\alpha)$. 则函数族 $\{(y^\alpha, Y^\alpha, z^\alpha, Z^\alpha); \alpha \in R\}$ 在 $M^2\left(0, T; R^{n+m+n\times l+m\times d}\right)$ 中关于 α 是连续的.*

证明: 为记号方便,我们仅证方程 (3.1) 在 $\alpha = 0$ 时连续. 为证 $\alpha \to 0$ 时, $(y^\alpha, Y^\alpha, z^\alpha, Z^\alpha, y_T^\alpha, Y_0^\alpha)$ 在 $M^2\left(0, T; R^{n+m+n\times l+m\times d}\right) \times L^2\left(\Omega, \mathcal{F}_T, P; R^n\right) \times L^2\left(\Omega, \mathcal{F}_0, P; R^m\right)$ 中,收敛于 $(y^0, Y^0, z^0, Z^0, y_T^0, Y_0^0)$,

令 $\widehat{U}_t = U_t^\alpha - U_t^0 = \left(\widehat{y}_t, \widehat{Y}_t, \widehat{z}_t, \widehat{Z}_t\right) = (y_t^\alpha - y_t^0, Y_t^\alpha - Y_t^0, z_t^\alpha - z_t^0, Z_t^\alpha - Z_t^0)$. 则

$$\begin{aligned} d\widehat{y}_t &= \left(f_\alpha\left(t, U_t^\alpha\right) - f_0\left(t, U_t^0\right)\right) dt + \left(g_\alpha\left(t, U_t^\alpha\right) - g_0\left(t, U_t^0\right)\right) dW_t - \widehat{z}_t dB_t, \\ d\widehat{Y}_t &= \left(F_\alpha\left(t, U_t^\alpha\right) - F_0\left(t, U_t^0\right)\right) dt + \left(G_\alpha\left(t, U_t^\alpha\right) - G_0\left(t, U_t^0\right)\right) dB_t + \widehat{Z}_t dW_t, \\ \widehat{y}_0 &= \Psi_\alpha\left(Y_0^\alpha\right) - \Psi_0\left(Y_0^0\right), \\ \widehat{Y}_T &= \Phi_\alpha\left(y_T^\alpha\right) - \Phi_0\left(y_T^0\right). \end{aligned}$$

由 (H3)-(H5), 对 $(\widehat{y}_t, \widehat{z}_t)$ 和 $\left(\widehat{Y}_t, \widehat{Z}_t\right)$ 利用类似于引理 2.5 技巧,可以得到

$$\begin{aligned} &\mathbf{E}\left|\widehat{y}_T\right|^2 + \mathbf{E}\int_0^T \left(\left|\widehat{y}_t\right|^2 + \left|\widehat{z}_t\right|^2\right) dt \\ \le\ & C_1 \mathbf{E}\int_0^T \left(\left|\widehat{Y}_t\right|^2 + \left|\widehat{Z}_t\right|^2\right) dt + C_1 \mathbf{E}\left|\widehat{Y}_0\right|^2 + C_1 \mathbf{E}\left|\Psi_\alpha\left(Y_0^0\right) - \Psi_0\left(Y_0^0\right)\right|^2 \\ & + C_1 \mathbf{E}\int_0^T \left|f_\alpha\left(t, U_t^0\right) - f_0\left(t, U_t^0\right)\right|^2 dt + C_1 \mathbf{E}\int_0^T \left|g_\alpha\left(t, U_t^0\right) - g_0\left(t, U_t^0\right)\right|^2 dt, \end{aligned}$$

和

$$
\begin{aligned}
&\mathbf{E}\left|\widehat{Y}_0\right|^2+\mathbf{E}\int_0^T\left(\left|\widehat{Y}_t\right|^2+\left|\widehat{Z}_t\right|^2\right)dt\\
\leq\ & C_1\mathbf{E}\int_0^T\left(|\widehat{y}_t|^2+|\widehat{z}_t|^2\right)dt+C_1\mathbf{E}\left|\widehat{y}_T\right|^2+C_1\mathbf{E}\left|\Phi_\alpha\left(y_T^0\right)-\Phi_0\left(y_T^0\right)\right|^2\\
&+C_1\mathbf{E}\int_0^T\left|F_\alpha\left(t,U_t^0\right)-F_0\left(t,U_t^0\right)\right|^2dt+C_1\mathbf{E}\int_0^T\left|G_\alpha\left(t,U_t^0\right)-G_0\left(t,U_t^0\right)\right|^2dt,
\end{aligned}
$$

这里 C_1 是依赖于利普希茨常数 k,λ 及 T 的常数.
在 [0,T] 上对 $\left\langle H\widehat{y}_t,\widehat{Y}_t\right\rangle$ 应用 Itô 公式，可以得到

$$
\begin{aligned}
&\mathbf{E}\left\langle H\widehat{y}_T,\Phi_\alpha\left(y_T^\alpha\right)-\Phi_\alpha\left(y_T^0\right)\right\rangle+\mathbf{E}\left\langle H\widehat{y}_T,\Phi_\alpha\left(y_T^0\right)-\Phi_0\left(y_T^0\right)\right\rangle\\
&-\mathbf{E}\left\langle H^T\widehat{Y}_0,\Psi_\alpha\left(Y_0^\alpha\right)-\Psi_\alpha\left(Y_0^0\right)\right\rangle-\mathbf{E}\left\langle H^T\widehat{Y}_0,\Psi_\alpha\left(Y_0^0\right)-\Psi_0\left(Y_0^0\right)\right\rangle\\
=\ &\mathbf{E}\int_0^T\left\langle A_\alpha\left(t,U_t^\alpha\right)-A_\alpha\left(t,U_t^0\right),\widehat{U}\right\rangle dt\\
&+\mathbf{E}\int_0^T\left(\left\langle\widehat{Y}_t,H\left(f_\alpha\left(t,U_t^0\right)-f_0\left(t,U_t^0\right)\right)\right\rangle+\left\langle H\widehat{y}_t,F_\alpha\left(t,U_t^0\right)-F_0\left(t,U_t^0\right)\right\rangle\right)dt\\
&+\mathbf{E}\int_0^T\left(\left\langle\widehat{Z}_t,H\left(g_\alpha\left(t,U_t^0\right)-g_0\left(t,U_t^0\right)\right)\right\rangle+\left\langle H\widehat{z}_t,G_\alpha\left(t,U_t^0\right)-G_0\left(t,U_t^0\right)\right\rangle\right)dt,
\end{aligned}
$$

这里 $A_\alpha(t,U)=\left(H^TF_\alpha,Hf_\alpha,H^TG_\alpha,Hg_\alpha\right)(t,U)$.
从而由 (H1) 和 (H2)，可以得到

$$
\begin{aligned}
&\mu_1\mathbf{E}\int_0^T\left(|\widehat{y}_t|^2+|\widehat{z}_t|^2\right)dt+\mu_2\mathbf{E}\int_0^T\left(\left|\widehat{Y}_t\right|^2+\left|\hat{Z}_t\right|^2\right)dt+\beta_1\mathbf{E}\left|\widehat{y}_T\right|^2+\beta_2\mathbf{E}\left|\widehat{Y}_0\right|^2\\
\leq\ &\delta\mathbf{E}\left|\widehat{y}_T\right|^2+\delta\mathbf{E}\left|\widehat{Y}_0\right|^2+\delta\mathbf{E}\int_0^T\left|\widehat{U}_t\right|^2dt\\
&+C_2\mathbf{E}\left|\Psi_\alpha\left(Y_0^0\right)-\Psi_0\left(Y_0^0\right)\right|^2+C_2\mathbf{E}\left|\Phi_\alpha\left(y_T^0\right)-\Phi_0\left(y_T^0\right)\right|^2\\
&+C_2\mathbf{E}\int_0^T\left(\left|f_\alpha\left(t,U_t^0\right)-f_0\left(t,U_t^0\right)\right|^2+\left|g_\alpha\left(t,U_t^0\right)-g_0\left(t,U_t^0\right)\right|^2\right)dt\\
&+C_2\mathbf{E}\int_0^T\left(\left|F_\alpha\left(t,U_t^0\right)-F_0\left(t,U_t^0\right)\right|^2+\left|G_\alpha\left(t,U_t^0\right)-G_0\left(t,U_t^0\right)\right|^2\right)dt.
\end{aligned}
$$

其中 C_2 依赖于 H 和 δ 的常数， $\delta=\min\left(\frac{\mu_1\beta_1}{2(1+C_1)(\mu_1+\beta_1)},\frac{\mu_2\beta_2}{2(1+C_1)(\mu_2+\beta_2)}\right)$.

结合上面的三个估计，无论 $m \geq n$ 或 $m \leq n$, 我们都有

$$\begin{aligned}
&\mathbf{E}\int_0^T \left|\widehat{U}_t\right|^2 dt + \mathbf{E}\left|\widehat{y}_T\right|^2 + \mathbf{E}\left|\widehat{Y}_0\right|^2 \\
\leq\ & C\mathbf{E}\int_0^T \left(\left|f_\alpha\left(t,U_t^0\right) - f_0\left(t,U_t^0\right)\right|^2 + \left|g_\alpha\left(t,U_t^0\right) - g_0\left(t,U_t^0\right)\right|^2\right) dt \\
&+C\mathbf{E}\int_0^T \left(\left|F_\alpha\left(t,U_t^0\right) - F_0\left(t,U_t^0\right)\right|^2 + \left|G_\alpha\left(t,U_t^0\right) - G_0\left(t,U_t^0\right)\right|^2\right) dt \\
&+C\mathbf{E}\left|\Phi_\alpha\left(y_T^0\right) - \Phi_0\left(y_T^0\right)\right|^2 + C\mathbf{E}\left|\Psi_\alpha\left(Y_0^0\right) - \Psi_0\left(Y_0^0\right)\right|^2,
\end{aligned}$$

这里 C 依赖于 C_1, C_2, μ_1, μ_2, β_1 及 β_2 的常数. 从而由 (H5) 可以得到：当 $\alpha \to 0$ 时，在 $M^2\left(0,T;R^{n+m+n\times l+m\times d}\right)\times L^2\left(\Omega,\mathcal{F}_T,P;R^n\right)\times L^2\left(\Omega,\mathcal{F}_0,P;R^m\right)$ 中 $(y^\alpha, Y^\alpha, z^\alpha, Z^\alpha, y_T^\alpha, Y_0^\alpha)$ 收敛于 $(y^0, Y^0, z^0, Z^0, y_T^0, Y_0^0)$.

下面，讨论方程 (2.1) 解关于参数的可微性. 我们有

定理 3.2 *令 $\{f_\alpha, g_\alpha, F_\alpha, G_\alpha, \Psi_\alpha, \Phi_\alpha; \alpha\in R\}$ 为一族正倒向重随机微分方程 (3.1)，满足 (H1)-(H4) 和如下假设*

$$(H6)\begin{cases}\text{(1) 对给个}\alpha\in R,\ \{f_\alpha(t,U), g_\alpha(t,U), F_\alpha(t,U), G_\alpha(t,U), \Psi_\alpha(Y), \Phi_\alpha(y)\}\\ \text{分别关于}\ (U,Y,y)\ \text{连续可微且导数一致有界;}\\ \text{(2) 函数族}\ \{f_\alpha, g_\alpha, F_\alpha, G_\alpha, \Psi_\alpha, \Phi_\alpha; \alpha\in R\}\ \text{关于}\alpha\text{是可微的.}\end{cases}$$

其解记为 $(y^\alpha, Y^\alpha, z^\alpha, Z^\alpha)$.

那么函数族 $\{(y^\alpha, Y^\alpha, z^\alpha, Z^\alpha); \alpha\in R\}$ 在 $M^2\left(0,T;R^{n+m+n\times l+m\times d}\right)$ 中关于 α 是可微的，且在 $\alpha=\alpha_0$ 处的导数是如下线性正倒向重随机微分方程的解 $(\delta_\alpha y^{\alpha_0}, \delta_\alpha Y^{\alpha_0}, \delta_\alpha z^{\alpha_0}, \delta_\alpha Z^{\alpha_0})$

$$\begin{cases}
d\delta_\alpha y_t^{\alpha_0} = [\delta_\alpha f_{\alpha_0}(t,U_t^{\alpha_0}) + \delta_y f_{\alpha_0}(t,U_t^{\alpha_0})(\delta_\alpha y_t^{\alpha_0}) + \delta_Y f_{\alpha_0}(t,U_t^{\alpha_0})(\delta_\alpha Y_t^{\alpha_0}) \\
\qquad +\delta_z f_{\alpha_0}(t,U_t^{\alpha_0})(\delta_\alpha z_t^{\alpha_0}) + \delta_Z f_{\alpha_0}(t,U_t^{\alpha_0})(\delta_\alpha Z_t^{\alpha_0})]dt \\
\qquad +[\delta_\alpha g_{\alpha_0}(t,U_t^{\alpha_0}) + \delta_y g_{\alpha_0}(t,U_t^{\alpha_0})(\delta_\alpha y_t^{\alpha_0}) + \delta_Y g_{\alpha_0}(t,U_t^{\alpha_0})(\delta_\alpha Y_t^{\alpha_0}) \\
\qquad +\delta_z g_{\alpha_0}(t,U_t^{\alpha_0})(\delta_\alpha z_t^{\alpha_0}) + \delta_Z g_{\alpha_0}(t,U_t^{\alpha_0})(\delta_\alpha Z_t^{\alpha_0})]dW_t - (\delta_\alpha z_t^{\alpha_0})dB_t, \\
d\delta_\alpha Y_t^{\alpha_0} = [\delta_\alpha F_{\alpha_0}(t,U_t^{\alpha_0}) + \delta_y F_{\alpha_0}(t,U_t^{\alpha_0})(\delta_\alpha y_t^{\alpha_0}) + \delta_Y F_{\alpha_0}(t,U_t^{\alpha_0})(\delta_\alpha Y_t^{\alpha_0}) \\
\qquad +\delta_z F_{\alpha_0}(t,U_t^{\alpha_0})(\delta_\alpha z_t^{\alpha_0}) + \delta_Z F_{\alpha_0}(t,U_t^{\alpha_0})(\delta_\alpha Z_t^{\alpha_0})]dt \\
\qquad +[\delta_\alpha G_{\alpha_0}(t,U_t^{\alpha_0}) + \delta_y G_{\alpha_0}(t,U_t^{\alpha_0})(\delta_\alpha y_t^{\alpha_0}) + \delta_Y G_{\alpha_0}(t,U_t^{\alpha_0})(\delta_\alpha Y_t^{\alpha_0}) \\
\qquad +\delta_z G_{\alpha_0}(t,U_t^{\alpha_0})(\delta_\alpha z_t^{\alpha_0}) + \delta_Z G_{\alpha_0}(t,U_t^{\alpha_0})(\delta_\alpha Z_t^{\alpha_0})]dB_t + (\delta_\alpha Z_t^{\alpha_0})dW_t, \\
\delta_\alpha y_0^{\alpha_0} = \delta_Y \Psi_{\alpha_0}(Y_0^{\alpha_0})(\delta_\alpha Y_0^{\alpha_0}) + \delta_\alpha \Psi_{\alpha_0}(Y_0^{\alpha_0}), \\
\delta_\alpha Y_T^{\alpha_0} = \delta_y \Phi_{\alpha_0}(y_T^{\alpha_0})(\delta_\alpha y_T^{\alpha_0}) + \delta_\alpha \Phi_{\alpha_0}(y_T^{\alpha_0}).
\end{cases} \tag{3.2}$$

证明: 为记号方便, 我们假设 $m=n=d=l=1$, 且只证明在 $\alpha_0=0$ 处的可微性. 令

$$\Delta_\alpha y_t=\frac{y_t^\alpha-y_t^0}{\alpha},\quad \Delta_\alpha Y_t=\frac{Y_t^\alpha-Y_t^0}{\alpha},\quad \Delta_\alpha z_t=\frac{z_t^\alpha-z_t^0}{\alpha},\quad \Delta_\alpha Z_t=\frac{Z_t^\alpha-Z_t^0}{\alpha}.$$

则

$$\begin{aligned}
d\Delta_\alpha y_t &= \frac{f_\alpha\left(t,U_t^\alpha\right)-f_0\left(t,U_t^0\right)}{\alpha}dt+\frac{g_\alpha\left(t,U_t^\alpha\right)-g_0\left(t,U_t^0\right)}{\alpha}dW_t-\Delta_\alpha z_t dB_t,\\
d\Delta_\alpha Y_t &= \frac{F_\alpha\left(t,U_t^\alpha\right)-F_0\left(t,U_t^0\right)}{\alpha}dt+\frac{G_\alpha\left(t,U_t^\alpha\right)-G_0\left(t,U_t^0\right)}{\alpha}dB_t+\Delta_\alpha Z_t dW_t,\\
\Delta_\alpha y_0 &= \frac{\Psi_\alpha\left(Y_0^\alpha\right)-\Psi_0\left(Y_0^0\right)}{\alpha},\\
\Delta_\alpha Y_T &= \frac{\Phi_\alpha\left(y_T^\alpha\right)-\Phi_0\left(y_T^0\right)}{\alpha}.
\end{aligned}$$

我们可以把上面的方程写成

$$\begin{cases}
d\Delta_\alpha y_t=\bar{f}_\alpha\left(t,\Delta_\alpha y_t,\Delta_\alpha Y_t,\Delta_\alpha z_t,\Delta_\alpha Z_t\right)dt\\
\qquad\qquad +\bar{g}_\alpha\left(t,\Delta_\alpha y_t,\Delta_\alpha Y_t,\Delta_\alpha z_t,\Delta_\alpha Z_t\right)dW_t-\Delta_\alpha z_t dB_t,\\
d\Delta_\alpha Y_t=\bar{F}_\alpha\left(t,\Delta_\alpha y_t,\Delta_\alpha Y_t,\Delta_\alpha z_t,\Delta_\alpha Z_t\right)dt\\
\qquad\qquad +\bar{G}_\alpha\left(t,\Delta_\alpha y_t,\Delta_\alpha Y_t,\Delta_\alpha z_t,\Delta_\alpha Z_t\right)dB_t+\Delta_\alpha Z_t dW_t,\\
\Delta_\alpha y_0=\frac{\Psi_\alpha\left(Y_0^\alpha\right)-\Psi_\alpha\left(Y_0^0\right)}{Y_0^\alpha-Y_0^0}\Delta_\alpha Y_0+\frac{\Psi_\alpha\left(Y_0^0\right)-\Psi_0\left(Y_0^0\right)}{\alpha},\\
\Delta_\alpha Y_T=\frac{\Phi_\alpha\left(y_T^\alpha\right)-\Phi_\alpha\left(y_T^0\right)}{y_T^\alpha-y_T^0}\Delta_\alpha y_T+\frac{\Phi_\alpha\left(y_T^0\right)-\Phi_0\left(y_T^0\right)}{\alpha},
\end{cases}\tag{3.3}$$

这里

$$\bar{l}_\alpha\left(t,\Delta_\alpha y_t,\Delta_\alpha Y_t,\Delta_\alpha z_t,\Delta_\alpha Z_t\right)=A_\alpha^l(t)\Delta_\alpha y_t+B_\alpha^l(t)\Delta_\alpha Y_t+C_\alpha^l(t)\Delta_\alpha z_t+D_\alpha^l(t)\Delta_\alpha Z_t+E_\alpha^l(t),$$

$l=f,g,F,G$, 且

$$\begin{aligned}
A_\alpha^l(t) &= \begin{cases}\frac{l_\alpha\left(t,y^\alpha,Y^\alpha,z^\alpha,Z^\alpha\right)-l_\alpha\left(t,y^0,Y^\alpha,z^\alpha,Z^\alpha\right)}{y_t^\alpha-y_t^0}, & y_t^\alpha-y_t^0\neq 0,\\ 0, & \text{其他};\end{cases}\\
B_\alpha^l(t) &= \begin{cases}\frac{l_\alpha\left(t,y^0,Y^\alpha,z^\alpha,Z^\alpha\right)-l_\alpha\left(t,y^0,Y^0,z^\alpha,Z^\alpha\right)}{Y_t^\alpha-Y_t^0}, & Y_t^\alpha-Y_t^0\neq 0,\\ 0, & \text{其他};\end{cases}\\
C_\alpha^l(t) &= \begin{cases}\frac{l_\alpha\left(t,y^0,Y^0,z^\alpha,Z^\alpha\right)-l_\alpha\left(t,y^0,Y^0,z^0,Z^\alpha\right)}{z_t^\alpha-z_t^0}, & z_t^\alpha-z_t^0\neq 0,\\ 0, & \text{其他};\end{cases}\\
D_\alpha^l(t) &= \begin{cases}\frac{l_\alpha\left(t,y^0,Y^0,z^0,Z^\alpha\right)-l_\alpha\left(t,y^0,Y^0,z^0,Z^0\right)}{Z_t^\alpha-Z_t^0}, & Z_t^\alpha-Z_t^0\neq 0,\\ 0, & \text{其他};\end{cases}\\
E_\alpha^l(t) &= \frac{l_\alpha\left(t,y^0,Y^0,z^0,Z^0\right)-l_0\left(t,y^0,Y^0,z^0,Z^0\right)}{\alpha}.
\end{aligned}$$

显然 (H6) 包含 (H5), 则从定理 3.1, 当 $\alpha \to 0$ 时, 在 $M^2(0,T)$ 中 $(y^\alpha, Y^\alpha, z^\alpha, Z^\alpha)$ 收敛于 (y^0, Y^0, z^0, Z^0) 由 (H6), 可以得到

$$\begin{aligned}
\lim_{\alpha\to 0} A_\alpha^l(t) &= \delta_y l_0\left(t, U_t^0\right), \\
\lim_{\alpha\to 0} B_\alpha^l(t) &= \delta_Y l_0\left(t, U_t^0\right), \\
\lim_{\alpha\to 0} C_\alpha^l(t) &= \delta_z l_0\left(t, U_t^0\right), \\
\lim_{\alpha\to 0} D_\alpha^l(t) &= \delta_Z l_0\left(t, U_t^0\right), \\
\lim_{\alpha\to 0} E_\alpha^l(t) &= \delta_\alpha l_0\left(t, U_t^0\right),
\end{aligned}$$

且

$$\begin{aligned}
\lim_{\alpha\to 0} \frac{\Psi_\alpha\left(Y_0^\alpha\right) - \Psi_\alpha\left(Y_0^0\right)}{Y_0^\alpha - Y_0^0} &= \delta_Y \Psi_0\left(Y_0^0\right), \\
\lim_{\alpha\to 0} \frac{\Psi_\alpha\left(Y_0^0\right) - \Psi_0\left(Y_0^0\right)}{\alpha} &= \delta_\alpha \Psi_0\left(Y_0^0\right), \\
\lim_{\alpha\to 0} \frac{\Phi_\alpha\left(y_T^\alpha\right) - \Phi_\alpha\left(y_T^0\right)}{y_T^\alpha - y_T^0} &= \delta_y \Phi_0\left(y_T^0\right), \\
\lim_{\alpha\to 0} \frac{\Phi_\alpha\left(y_T^0\right) - \Phi_0\left(y_T^0\right)}{\alpha} &= \delta_\alpha \Phi_0\left(y_T^0\right).
\end{aligned}$$

从而得到

$$\begin{aligned}
\lim_{\alpha\to 0} \bar{l}_\alpha\left(t, \Delta_\alpha y_t, \Delta_\alpha Y_t, \Delta_\alpha z_t, \Delta_\alpha Z_t\right) &= \delta_\alpha l_0\left(t, U_t^0\right) + \delta_y l_0\left(t, U_t^0\right)\delta_\alpha y_t^0 + \delta_Y l_0\left(t, U_t^0\right)\delta_\alpha Y_t^0 \\
&\quad + \delta_z l_0\left(t, U_t^0\right)\delta_\alpha z_t^0 + \delta_Z l_0\left(t, U_t^0\right)\delta_\alpha Z_t^0.
\end{aligned}$$

由于 $\{f_\alpha, g_\alpha, F_\alpha, G_\alpha, \Psi_\alpha, \Phi_\alpha; \alpha \in R\}$ 满足 (H1)-(H4) 和 (H6), 我们容易验证方程 (3.2) 和 (3.3) 满足 (H1)-(H4). 当 $\alpha \to 0$ 时,在 $M^2(0,T)$ 中 (3.3) 的唯一解 $(\Delta_\alpha y_t, \Delta_\alpha Y_t, \Delta_\alpha z_t, \Delta_\alpha Z_t)$ 收敛到 (3.2) 的唯一解 $(\delta_\alpha y^0, \delta_\alpha Y^0, \delta_\alpha z^0, \delta_\alpha Z^0)$ (当 $\alpha_0 = 0$ 时). 结论得证.

第四章　正倒向随机微分方程解存在唯一性定理的另一种证法

§4.1　基本估计

我们定义

$\|u(\cdot)\| \doteq \left(\mathbf{E}\int_0^T |u(s)|^2 ds\right) < \infty$. 对于 $\forall\lambda \in R$,

$\|u(\cdot)\|_\lambda \doteq \left(\mathbf{E}\int_0^T \exp(-\lambda s)|u(s)|^2 ds\right) < \infty$.

显然，这两个范数 $\|\cdot\|_\lambda$，和 $\|\cdot\|$ 是等价的.

如下单调性假设是我们的主要假设:

(A1) 存在 $\lambda_1, \lambda_2 \in R$, 满足对所有 $t, y, y_1, y_2, Y, Y_1, Y_2, z, Z$, a.s.

$$\langle f(t, y_1, Y, z, Z) - f(t, y_2, Y, z, Z), y_1 - y_2\rangle \le \lambda_1 |y_1 - y_2|^2,$$
$$\langle F(t, y, Y_1, z, Z) - F(t, y, Y_2, z, Z), Y_1 - Y_2\rangle \le \lambda_2 |Y_1 - Y_2|^2.$$

(A2) 函数 f 关于 (Y, z, Z) 是一致利普希茨连续， F 关于 (y, z, Z) 是一致利普希茨连续. 换句话说，存在 $k_i > 0, i = 1,2,3,4,5,6$ 满足所有 $t, y, y_1, y_2, Y, Y_1, Y_2, z_1, z_2, Z_1, Z_2$, a.s.

$$|f(t, y, Y_1, z_1, Z_1) - f(t, y, Y_2, z_2, Z_2)| \le k_1 |Y_1 - Y_2| + k_2 |z_1 - z_2| + k_3 |Z_1 - Z_2|,$$
$$|F(t, y_1, Y, z_1, Z_1) - F(t, y_2, Y, z_2, Z_2)| \le k_4 |y_1 - y_2| + k_5 |z_1 - z_2| + k_6 |Z_1 - Z_2|.$$

(A3) 函数 g 是关于 (y, Y, z, Z) 是一致利普希茨连续，G 关于 (y, Y, z, Z) 是一致利普希茨连续. 也就是说, 存在 $k_i, i = 7,8,9,10,11,12, 0 < \alpha < 1$, 对所有 $t, y_1, y_2, Y_1, Y_2, z_1, z_2, Z_1, Z_2$, a.s.

$$|g(t, y_1, Y_1, z_1, Z_1) - g(t, y_2, Y_2, z_2, Z_2)|^2$$
$$\le k_7^2 |y_1 - y_2|^2 + k_8^2 |Y_1 - Y_2|^2 + \alpha |z_1 - z_2|^2 + k_9^2 |Z_1 - Z_2|^2,$$
$$|G(t, y_1, Y_1, z_1, Z_1) - G(t, y_2, Y_2, z_2, Z_2)|^2$$
$$\le k_{10}^2 |y_1 - y_2|^2 + k_{11}^2 |Y_1 - Y_2|^2 + k_{12}^2 |z_1 - z_2|^2 + \alpha |Z_1 - Z_2|^2.$$

(A4) 函数 Ψ 关于 Y 是一致利普希茨连续， Φ 关于 y 是一致利普希茨连续. 也就是说，存在 $k_i, i = 13, 14$, 对所有 y_1, y_2, Y_1, Y_2, a.s.

$$|\Psi(Y_1) - \Psi(Y_2)| \le k_{13} |Y_1 - Y_2|,$$
$$|\Phi(y_1) - \Phi(y_2)| \le k_{14} |y_1 - y_2|.$$

引理 4.1 在假设 $(A1)-(A4)$ 下，令 $(y_i(\cdot), z_i(\cdot))$ 是方程 *(2.1)* 中正向方程的解，相应地方程 *(2.1)* 中倒向方程的解为 $(Y(\cdot), Z(\cdot)) = (Y_i(\cdot), Z_i(\cdot)) \in M^2(0,T;R^m) \times M^2(0,T;R^{m\times d})$, $i=1,2$. 那么对于所有 $\lambda \in R, C_1, C_2, C_3 > 0$,

$$\begin{aligned}
&exp(-\lambda t)\mathbf{E}|y_1(t)-y_2(t)|^2 + \bar{\lambda}_1\int_0^t \exp(-\lambda s)\mathbf{E}|y_1(s)-y_2(s)|^2 ds \\
&\quad +(1-\alpha-k_2C_2)\int_0^t \exp(-\lambda s)\mathbf{E}|z_1(s)-z_2(s)|^2 ds \\
&\leq k_{13}^2\mathbf{E}|Y_1(0)-Y_2(0)|^2 \\
&\quad +(k_8^2+k_1C_1)\int_0^t \exp(-\lambda s)\mathbf{E}|Y_1(s)-Y_2(s)|^2 ds \\
&\quad +(k_9^2+k_3C_3)\int_0^t \exp(-\lambda s)\mathbf{E}|Z_1(s)-Z_2(s)|^2 ds,
\end{aligned} \tag{4.1}$$

这里 $\bar{\lambda}_1 \doteq \lambda - 2\lambda_1 - k_1C_1^{-1} - k_2C_2^{-1} - k_3C_3^{-1} - k_7^2$. 从而，

$$\begin{aligned}
&exp(-\lambda t)\mathbf{E}|y_1(t)-y_2(t)|^2 \\
&\quad +(1-\alpha-k_2C_2)\int_0^t \exp\left(-\bar{\lambda}_1(t-s)\right)\exp(-\lambda s)\mathbf{E}|z_1(s)-z_2(s)|^2 ds \\
&\leq k_{13}^2\exp\left(-\bar{\lambda}_1 t\right)\mathbf{E}|Y_1(0)-Y_1(0)|^2 \\
&\quad +(k_8^2+k_1C_1)\int_0^t \exp\left(-\bar{\lambda}_1(t-s)\right)\exp(-\lambda s)\mathbf{E}|Y_1(s)-Y_2(s)|^2 ds \\
&\quad +(k_9^2+k_3C_3)\int_0^t \exp\left(-\bar{\lambda}_1(t-s)\right)\exp(-\lambda s)\mathbf{E}|Z_1(s)-Z_2(s)|^2 ds.
\end{aligned} \tag{4.2}$$

注 4.2 若满足 $0 < C_2 < \frac{1}{k_2}(1-\alpha)$, 那么，我们由 *(4.2)* 可以得到下面的不等式

$$\begin{aligned}
&\|y_1(\cdot)-y_2(\cdot)\|_\lambda^2 \\
&\leq \tfrac{1-\exp\left(-\bar{\lambda}_1 T\right)}{\bar{\lambda}_1}\left[\begin{array}{c} k_{13}^2\mathbf{E}|Y_1(0)-Y_2(0)|^2 + (k_8^2+k_1C_1)\|Y_1(\cdot)-Y_2(\cdot)\|_\lambda^2 \\ +(k_9^2+k_3C_3)\|Z_1(\cdot)-Z_2(\cdot)\|_\lambda^2 \end{array}\right].
\end{aligned} \tag{4.3}$$

若还有 $\bar{\lambda}_1 \geq 0$, 我们由 *(4.1)* 可得

$$\begin{aligned}
&\|z_1(\cdot)-z_2(\cdot)\|_\lambda^2 \\
&\leq \tfrac{1}{1-\alpha-k_2C_2}\left[\begin{array}{c} k_{13}^2\mathbf{E}|Y_1(0)-Y_2(0)|^2 + (k_8^2+k_1C_1)\|Y_1(\cdot)-Y_2(\cdot)\|_\lambda^2 \\ +(k_9^2+k_3C_3)\|Z_1(\cdot)-Z_2(\cdot)\|_\lambda^2 \end{array}\right],
\end{aligned} \tag{4.4}$$

$$\begin{aligned}
&exp(-\lambda T)\mathbf{E}|y_1(T)-y_2(T)|^2 \\
&\leq k_{13}^2\mathbf{E}|Y_1(0)-Y_2(0)|^2 + (k_8^2+k_1C_1)\|Y_1(\cdot)-Y_2(\cdot)\|_\lambda^2 + (k_9^2+k_3C_3)\|Z_1(\cdot)-Z_2(\cdot)\|_\lambda^2.
\end{aligned} \tag{4.5}$$

证明: 在 $[0,t]$ 上对 $\exp(-\lambda s)|y_1(s)-y_2(s)|^2$ 应用 Itô 公式，得到

$$\begin{aligned}
&\exp(-\lambda t)\mathbf{E}|y_1(t)-y_2(t)|^2 \\
&=\mathbf{E}|y_1(0)-y_2(0)|^2 \\
&-\lambda\int_0^t \exp(-\lambda s)\mathbf{E}|y_1(s)-y_2(s)|^2 ds
\end{aligned}$$

$$
\begin{aligned}
&+\mathbf{E}\int_0^t\exp(-\lambda s)\,2(y_1(s)-y_2(s)) \\
&\quad [f(s,y_1(s),Y_1(s),z_1(s),Z_1(s))-f(s,y_2(s),Y_2(s),z_2(s),Z_2(s))]\,ds \\
&+\mathbf{E}\int_0^t\exp(-\lambda s)\,|g(t,y_1(s),Y_1(s),z_1(s),Z_1(s))-g(t,y_2(s),Y_2(s),z_2(s),Z_2(s))|^2\,ds \\
&-\mathbf{E}\int_0^t\exp(-\lambda s)\,|z_1(s)-z_2(s)|^2\,ds \\
&\le k_{13}^2\mathbf{E}\,|Y_1(0)-Y_2(0)|^2 \\
&-\lambda\int_0^t\exp(-\lambda s)\,\mathbf{E}\,|y_1(s)-y_2(s)|^2\,ds \\
&+2\lambda_1\int_0^t\exp(-\lambda s)\,\mathbf{E}\,|y_1(s)-y_2(s)|^2\,ds \\
&+\mathbf{E}\int_0^t\exp(-\lambda s)\left[\begin{array}{l} k_1C_1^{-1}|y_1-y_2|^2+k_1C_1|Y_1-Y_2|^2+k_2C_2^{-1}|y_1-y_2|^2 \\ +k_2C_2|z_1-z_2|^2+k_3C_3^{-1}|y_1-y_2|^2+k_3C_3|Z_1-Z_2|^2 \end{array}\right]ds \\
&+\mathbf{E}\int_0^t\exp(-\lambda s)\left[k_7^2|y_1-y_2|^2+k_8^2|Y_1-Y_2|^2+\alpha|z_1-z_2|^2+k_9^2|Z_1-Z_2|^2\right]ds \\
&-\mathbf{E}\int_0^t\exp(-\lambda s)\,|z_1(s)-z_2(s)|^2\,ds,
\end{aligned}
$$

$$
\begin{aligned}
&\exp(-\lambda t)\,\mathbf{E}\,|y_1(t)-y_2(t)|^2+\bar{\lambda}_1\int_0^t\exp(-\lambda s)\,\mathbf{E}\,|y_1(s)-y_2(s)|^2\,ds \\
&+(1-\alpha-k_2C_2)\int_0^t\exp(-\lambda s)\,\mathbf{E}\,|z_1(s)-z_2(s)|^2\,ds \\
&\le k_{13}^2\mathbf{E}\,|Y_1(0)-Y_2(0)|^2 \\
&\quad+(k_8^2+k_1C_1)\int_0^t\exp(-\lambda s)\,\mathbf{E}\,|Y_1(s)-Y_2(s)|^2\,ds \\
&\quad+(k_9^2+k_3C_3)\int_0^t\exp(-\lambda s)\,\mathbf{E}\,|Z_1(s)-Z_2(s)|^2\,ds,
\end{aligned}
$$

由 Gronwall 不等式:

$0\le g(t)\le\alpha(t)+\beta\int_0^t g(s)\,ds$, 则 $g(t)\le\int_0^t\alpha'(s)\exp(\beta(t-s))\,ds$.

我们有

$$
\begin{aligned}
&\exp(-\lambda t)\,\mathbf{E}\,|y_1(t)-y_2(t)|^2 \\
&+(1-\alpha-k_2C_2)\int_0^t\exp\left(-\bar{\lambda}_1(t-s)\right)\exp(-\lambda s)\,\mathbf{E}\,|z_1(s)-z_2(s)|^2\,ds \\
&\le k_{13}^2\exp\left(-\bar{\lambda}_1 t\right)\mathbf{E}\,|Y_1(0)-Y_2(0)|^2 \\
&\quad+(k_8^2+k_1C_1)\int_0^t\exp\left(-\bar{\lambda}_1(t-s)\right)\exp(-\lambda s)\,\mathbf{E}\,|Y_1(s)-Y_2(s)|^2\,ds \\
&\quad+(k_9^2+k_3C_3)\int_0^t\exp\left(-\bar{\lambda}_1(t-s)\right)\exp(-\lambda s)\,\mathbf{E}\,|Z_1(s)-Z_2(s)|^2\,ds,
\end{aligned}
$$

如果 $(1-\alpha-k_2C_2)>0$, 则

$$
\begin{aligned}
&\int_0^T\exp(-\lambda t)\,\mathbf{E}\,|y_1(t)-y_2(t)|^2\,dt \\
&\le k_{13}^2\int_0^T\exp\left(-\bar{\lambda}_1 t\right)\mathbf{E}\,|Y_1(0)-Y_2(0)|^2\,dt \\
&\quad+(k_8^2+k_1C_1)\int_0^T\int_0^t\exp\left(-\bar{\lambda}_1(t-s)\right)\exp(-\lambda s)\,\mathbf{E}\,|Y_1(s)-Y_2(s)|^2\,dsdt \\
&\quad+(k_9^2+k_3C_3)\int_0^T\int_0^t\exp\left(-\bar{\lambda}_1(t-s)\right)\exp(-\lambda s)\,\mathbf{E}\,|Z_1(s)-Z_2(s)|^2\,dsdt.
\end{aligned}
$$

(a)If $\bar{\lambda}_1 < 0$,

$$\int_0^T \int_0^t \exp\left(-\bar{\lambda}_1 (t-s)\right) \exp(-\lambda s) \mathbf{E} |Y_1(s) - Y_2(s)|^2 dsdt$$
$$\leq \int_0^T \exp\left(-\bar{\lambda}_1 t\right) \int_0^T \exp(-\lambda s) \mathbf{E} |Y_1(s) - Y_2(s)|^2 dsdt$$
$$= \frac{1-\exp\left(-\bar{\lambda}_1 T\right)}{\bar{\lambda}_1} \|Y_1(\cdot) - Y_2(\cdot)\|_\lambda^2,$$

(b)If $\bar{\lambda}_1 > 0$,

$$\int_0^T \int_0^t \exp\left(-\bar{\lambda}_1 (t-s)\right) \exp(-\lambda s) \mathbf{E} |Y_1(s) - Y_2(s)|^2 dsdt$$
$$= \int_0^T \int_s^T \exp\left(-\bar{\lambda}_1 (t-s)\right) \exp(-\lambda s) \mathbf{E} |Y_1(s) - Y_2(s)|^2 dtds$$
$$= \int_0^T \exp\left(\bar{\lambda}_1 s\right) \exp(-\lambda s) \mathbf{E} |Y_1(s) - Y_2(s)|^2 \int_s^T \exp\left(-\bar{\lambda}_1 t\right) dtds$$
$$= \int_0^T \exp\left(\bar{\lambda}_1 s\right) \exp(-\lambda s) \mathbf{E} |Y_1(s) - Y_2(s)|^2 \frac{\exp\left(-\bar{\lambda}_1 s\right) - \exp\left(-\bar{\lambda}_1 T\right)}{\bar{\lambda}_1} ds$$
$$= \int_0^T \frac{1-\exp\left(\bar{\lambda}_1 s\right)\exp\left(-\bar{\lambda}_1 T\right)}{\bar{\lambda}_1} \exp(-\lambda s) \mathbf{E} |Y_1(s) - Y_2(s)|^2 ds$$
$$\leq \frac{1-\exp\left(-\bar{\lambda}_1 T\right)}{\bar{\lambda}_1} \int_0^T \exp(-\lambda s) \mathbf{E} |Y_1(s) - Y_2(s)|^2 ds$$
$$= \frac{1-\exp\left(-\bar{\lambda}_1 T\right)}{\bar{\lambda}_1} \|Y_1(\cdot) - Y_2(\cdot)\|_\lambda^2.$$

则

$$\|y_1(\cdot) - y_2(\cdot)\|_\lambda^2$$
$$\leq \frac{1-\exp\left(-\bar{\lambda}_1 T\right)}{\bar{\lambda}_1} \left[\begin{array}{c} k_{13}^2 \mathbf{E} |Y_1(0) - Y_2(0)|^2 + (k_8^2 + k_1 C_1) \|Y_1(\cdot) - Y_2(\cdot)\|_\lambda^2 \\ + (k_9^2 + k_3 C_3) \|Z_1(\cdot) - Z_2(\cdot)\|_\lambda^2, \end{array} \right]$$

若 $\bar{\lambda}_1 \geq 0, 1 - \alpha - k_2 C_2 > 0$，则

$$\|z_1(\cdot) - z_2(\cdot)\|_\lambda^2$$
$$\leq \frac{1}{1-\alpha-k_2 C_2} \left[\begin{array}{c} k_{13}^2 \mathbf{E} |Y_1(0) - Y_2(0)|^2 + (k_8^2 + k_1 C_1) \|Y_1(\cdot) - Y_2(\cdot)\|_\lambda^2 \\ + (k_9^2 + k_3 C_3) \|Z_1(\cdot) - Z_2(\cdot)\|_\lambda^2 \end{array} \right],$$

$$\exp(-\lambda T) \mathbf{E} |y_1(T) - y_2(T)|^2$$
$$\leq k_{13}^2 \mathbf{E} |Y_1(0) - Y_2(0)|^2 + (k_8^2 + k_1 C_1) \|Y_1(\cdot) - Y_2(\cdot)\|_\lambda^2 + (k_9^2 + k_3 C_3) \|Z_1(\cdot) - Z_2(\cdot)\|_\lambda^2$$

. 引理得证.

引理 4.3 *在假设 (A1)−(A4) 下. 令 $(Y_i(\cdot), Z_i(\cdot))$ 是方程 (2.1) 中倒向方程的解, 相应地方程 (2.1) 中正向方程的解为 $(y(\cdot), z(\cdot)) = (y_i(\cdot), z_i(\cdot)) \in M^2(0,T;R^m) \times M^2(0,T;R^{m\times d})$, $i = 1, 2$. 那么对于所有 $\lambda \in R, C_4, C_5, C_6 > 0$,*

$$\begin{aligned} & exp(-\lambda t) \mathbf{E} |Y_1(t) - Y_2(t)|^2 + \bar{\lambda}_2 \int_t^T \exp(-\lambda s) \mathbf{E} |Y_1(s) - Y_2(s)|^2 ds \\ & + (1 - \alpha - k_6 C_6) \int_t^T \exp(-\lambda s) \mathbf{E} |Z_1(s) - Z_2(s)|^2 ds \\ & \leq k_{14}^2 \exp(-\lambda T) \mathbf{E} |y_1(T) - y_2(T)|^2 \\ & + (k_{10}^2 + k_4 C_4) \int_t^T \exp(-\lambda s) \mathbf{E} |y_1(s) - y_2(s)|^2 ds \\ & + (k_{12}^2 + k_5 C_5) \int_t^T \exp(-\lambda s) \mathbf{E} |z_1(s) - z_2(s)|^2 ds, \end{aligned} \tag{4.6}$$

这里 $\bar{\lambda}_2 \doteq -(\lambda + 2\lambda_2 + k_4C_4^{-1} + k_5C_5^{-1} + k_6C_6^{-1} + k_{11}^2)$. 从而,

$$\begin{aligned}
&exp\left(-\lambda t\right)\mathbf{E}\left|Y_1(t) - Y_2(t)\right|^2 \\
&+ (1-\alpha-k_6C_6)\int_t^T \exp\left(-\bar{\lambda}_2(s-t)\right)\exp\left(-\lambda s\right)\mathbf{E}\left|Z_1(s) - Z_2(s)\right|^2 ds \\
&\le k_{14}^2 \exp\left(-\bar{\lambda}_2(T-t)\right)\exp\left(-\lambda T\right)\mathbf{E}\left|y_1(T) - y_1(T)\right|^2 \\
&+ (k_{10}^2 + k_4C_4)\int_t^T \exp\left(-\bar{\lambda}_2(T-t)\right)\exp\left(-\lambda s\right)\mathbf{E}\left|y_1(s) - y_2(s)\right|^2 ds \\
&+ (k_{12}^2 + k_5C_5)\int_t^T \exp\left(-\bar{\lambda}_2(T-t)\right)\exp\left(-\lambda s\right)\mathbf{E}\left|z_1(s) - z_2(s)\right|^2 ds.
\end{aligned} \tag{4.7}$$

注 4.4 若还有 $0 < C_6 < \frac{1}{k_6}(1-\alpha)$ 则, 我可由 *(4.7)* 得到如下不等式

$$\begin{aligned}
&\|Y_1(\cdot) - Y_2(\cdot)\|_\lambda^2 \\
&\le \frac{1-\exp(-\bar{\lambda}_2 T)}{\bar{\lambda}_2}\left[\begin{array}{c} k_{14}^2 \exp\left(-\lambda T\right)\mathbf{E}\left|y_1(T) - y_2(T)\right|^2 + (k_{10}^2 + k_4C_4)\|y_1(\cdot) - y_2(\cdot)\|_\lambda^2 \\ + (k_{12}^2 + k_5C_5)\|z_1(\cdot) - z_2(\cdot)\|_\lambda^2 \end{array}\right],
\end{aligned} \tag{4.8}$$

若 $\bar{\lambda}_2 \ge 0$, 我们由 *(4.6)* 可得

$$\begin{aligned}
&\|Z_1(\cdot) - Z_2(\cdot)\|_\lambda^2 \\
&\le \frac{1}{1-\alpha-k_6C_6}\left[\begin{array}{c} k_{14}^2 \exp\left(-\lambda T\right)\mathbf{E}\left|y_1(T) - y_2(T)\right|^2 + (k_{10}^2 + k_4C_4)\|y_1(\cdot) - y_2(\cdot)\|_\lambda^2 \\ + (k_{12}^2 + k_5C_5)\|z_1(\cdot) - z_2(\cdot)\|_\lambda^2 \end{array}\right],
\end{aligned} \tag{4.9}$$

$$\begin{aligned}
&\mathbf{E}\left|Y_1(0) - Y_2(0)\right|^2 \\
&\le k_{14}^2\mathbf{E}\left|y_1(T) - y_2(T)\right|^2 + (k_{10}^2 + k_4C_4)\|y_1(\cdot) - y_2(\cdot)\|_\lambda^2 + (k_{12}^2 + k_5C_5)\|z_1(\cdot) - z_2(\cdot)\|_\lambda^2.
\end{aligned} \tag{4.10}$$

证明从略.

最后，在结束本节前，我们定义两个映射 Γ_1, Γ_2.

注意到在正倒向重随机微分方程 (1.1) 的正向方程暗含了映射
$M_1 : M^2(0,T;R^m) \times M^2\left(0,T;R^{m\times d}\right) \to M^2(0,T;R^n) \times M^2\left(0,T;R^{n\times l}\right)$,
它对于每个 $(Y(\cdot), Z(\cdot)) \in M^2(0,T;R^m) \times M^2\left(0,T;R^{m\times d}\right)$ 对应 $M_1(Y(\cdot), Z(\cdot))$, 即正向方程的唯一解 $(y(\cdot), z(\cdot))$:
$y_t = \Psi(Y_0) + \int_0^t f(s, y_s, Y_s, z_s, Z_s)\,ds + \int_0^t g(s, y_s, Y_s, z_s, Z_s)\,dW_s - \int_0^t z_s dB_s$.

类似的，正倒向重随机微分方程 (1.1) 的倒向方程也暗含映射
$M_2 : M^2(0,T;R^n) \times M^2\left(0,T;R^{n\times l}\right) \to M^2(0,T;R^m) \times M^2\left(0,T;R^{m\times d}\right)$,
它对于每个 $(y(\cdot), z(\cdot)) \in M^2(0,T;R^n) \times M^2\left(0,T;R^{n\times l}\right)$ 对应 $M_2(y(\cdot), z(\cdot))$, 即倒向方程的唯一解 $(\bar{Y}(\cdot), \bar{Z}(\cdot))$:

$\bar{Y}_t = \Phi(y_T) + \int_t^T F(s, y_s, \bar{Y}_s, z_s, \bar{Z}_s)\, ds + \int_t^T G(s, y_s, \bar{Y}_s, z_s, \bar{Z}_s)\, dB_s + \int_t^T \bar{Z}_s dW_s.$

定义映射 Γ_1 为 $\Gamma_1 \doteq M_2 \circ M_1$ ，映射 Γ_2 为 $\Gamma_2 \doteq M_1 \circ M_2$. 可以证明 Γ_1 是 $M^2(0,T;R^m) \times M^2(0,T;R^{m\times d})$ 到自身的映射， Γ_2 是 $M^2(0,T;R^n) \times M^2(0,T;R^{n\times l})$ 到自身的映射.

对 $(Y_i(\cdot), Z_i(\cdot)) \in M^2(0,T;R^m) \times M^2(0,T;R^{m\times d})$，令 $(y_i(\cdot), z_i(\cdot)) \doteq M_1(Y_i(\cdot), Z_i(\cdot))$ 和 $(\bar{Y}_i(\cdot), \bar{Z}_i(\cdot)) \doteq \Gamma_1(Y_i(\cdot), Z_i(\cdot))$,$i = 1, 2.$

令

$$
\begin{aligned}
&a \doteq \|y_1(\cdot) - y_2(\cdot)\|_\lambda^2, b \doteq \|z_1(\cdot) - z_2(\cdot)\|_\lambda^2, \\
&c \doteq \|Y_1(\cdot) - Y_2(\cdot)\|_\lambda^2, d \doteq \|Z_1(\cdot) - Z_2(\cdot)\|_\lambda^2, \\
&\bar{c} \doteq \left\|\bar{Y}_1(\cdot) - \bar{Y}_2(\cdot)\right\|_\lambda^2, \bar{d} \doteq \left\|\bar{Z}_1(\cdot) - \bar{Z}_2(\cdot)\right\|_\lambda^2, \\
&A \doteq \exp(-\lambda T)\, \mathbf{E}\,|y_1(T) - y_2(T)|^2, B \doteq \mathbf{E}\,|Y_1(0) - Y_2(0)|^2,
\end{aligned}
$$

则由引理 4.1 和引理 4.3 ，我们有

$$
\begin{aligned}
&a \le \tfrac{1-\exp\left(-\bar{\lambda}_1 T\right)}{\bar{\lambda}_1}\left[k_{13}^2 B + (k_8^2 + k_1 C_1)c + (k_9^2 + k_3 C_3)\, d\right], \\
&b \le \tfrac{k_{13}^2 \exp\left(-\bar{\lambda}_1 T\right) B + \left(1 \vee \exp\left(-\bar{\lambda}_1 T\right)\right)\left[(k_8^2 + k_1 C_1)c + \left(k_9^2 + k_3 C_3\right) d\right]}{(1-\alpha-k_2 C_2)\left(1 \wedge \exp\left(-\bar{\lambda}_1 T\right)\right)}, \\
&\bar{c} \le \tfrac{1-\exp\left(-\bar{\lambda}_2 T\right)}{\bar{\lambda}_2}\left[k_{14}^2 A + (k_{10}^2 + k_4 C_4)\, a + (k_{12}^2 + k_5 C_5)\, b\right], \\
&\bar{d} \le \tfrac{k_{14}^2 \exp\left(-\bar{\lambda}_2 T\right) A + \left(1 \vee \exp\left(-\bar{\lambda}_2 T\right)\right)\left[(k_{10}^2 + k_4 C_4) a + \left(k_{12}^2 + k_5 C_5\right) b\right]}{(1-\alpha-k_6 C_6)\left(1 \wedge \exp\left(-\bar{\lambda}_2 T\right)\right)}, \\
&A \le \left(1 \vee \exp\left(-\bar{\lambda}_1 T\right)\right)\left[k_{13}^2 B + (k_8^2 + k_1 C_1)c + (k_9^2 + k_3 C_3)\, d\right], \\
&\bar{B} \le \left(1 \vee \exp\left(-\bar{\lambda}_2 T\right)\right)\left[k_{14}^2 A + (k_{10}^2 + k_4 C_4)\, a + (k_{12}^2 + k_5 C_5)\, b\right].
\end{aligned}
$$

进一步，当 $\bar{\lambda}_1 > 0, \bar{\lambda}_2 > 0$ 时，我们有

$$
\begin{aligned}
&A \le k_{13}^2 B + (k_8^2 + k_1 C_1)c + (k_9^2 + k_3 C_3)\, d, \\
&b \le \tfrac{k_{13}^2 \exp\left(-\bar{\lambda}_1 T\right) B + \left[(k_8^2 + k_1 C_1)c + \left(k_9^2 + k_3 C_3\right) d\right]}{(1-\alpha-k_2 C_2)}, \\
&\bar{B} \le k_{14}^2 A + (k_{10}^2 + k_4 C_4)\, a + (k_{12}^2 + k_5 C_5)\, b, \\
&\bar{d} \le \tfrac{k_{14}^2 \exp\left(-\bar{\lambda}_2 T\right) A + \left[(k_{10}^2 + k_4 C_4) a + \left(k_{12}^2 + k_5 C_5\right) b\right]}{(1-\alpha-k_6 C_6)}.
\end{aligned}
$$

注 4.5 *Feyel[F] 首先在常微分方程和随机微分方程解的存在唯一性证明中采用了等价范数技巧和压缩映射原理. 在下一节中，等价范数技巧在我们的单调性条件下证明正倒向重随机微分方程解的存在唯一性中起到关键的作用.*

§4.2 存在唯一性定理

定理 4.6 *假设 (A1)-(A4) 成立，则存在* $\varepsilon_0 > 0$, *它依赖于* $k_4, k_5, k_6, k_7, k_{10}, k_{11}, k_{12}, k_{14}, \lambda_1, \lambda_2, T$, *满足当* $k_1, k_3, k_8, k_9, k_{13} \in [0, \varepsilon_0)$ ，*正倒向重随机微分方程 (2.1) 存在唯一的适应解* (y, Y, z, Z) . *进一步，如果* $\lambda_1 + \lambda_2 < \frac{-\left(k_7^2 + k_{11}^2\right)}{2}$ ，*则存在* $\varepsilon_1 > 0$, *它依赖于* $k_4, k_5, k_6, k_7, k_{10}, k_{11}, k_{12}, k_{14}, \lambda_1, \lambda_2$ *且独立于* T, *满足当* $k_1, k_3, k_8, k_9, k_{13} \in [0, \varepsilon_1)$ ，*正倒向重随机微分方程 (2.1) 存在唯一的适应解.*

证明: 考虑映射 Γ_1. 我们有

$$
\begin{aligned}
&\bar{c} + \bar{d} + \bar{B} \\
&\le k_{14}^2 \left(\tfrac{1-\exp\left(-\bar{\lambda}_2 T\right)}{\bar{\lambda}_2} + \tfrac{\exp\left(-\bar{\lambda}_2 T\right)}{(1-\alpha-k_6 C_6)\left(1 \wedge \exp\left(-\bar{\lambda}_2 T\right)\right)} + \left(1 \vee \exp\left(-\bar{\lambda}_2 T\right)\right) \right) A \\
&\quad + \left(k_{10}^2 + k_4 C_4\right) \left(\tfrac{1-\exp\left(-\bar{\lambda}_2 T\right)}{\bar{\lambda}_2} + \tfrac{1 \vee \exp\left(-\bar{\lambda}_2 T\right)}{(1-\alpha-k_6 C_6)\left(1 \wedge \exp\left(-\bar{\lambda}_2 T\right)\right)} + \left(1 \vee \exp\left(-\bar{\lambda}_2 T\right)\right) \right) a \\
&\quad + \left(k_{12}^2 + k_5 C_5\right) \left(\tfrac{1-\exp\left(-\bar{\lambda}_2 T\right)}{\bar{\lambda}_2} + \tfrac{1 \vee \exp\left(-\bar{\lambda}_2 T\right)}{(1-\alpha-k_6 C_6)\left(1 \wedge \exp\left(-\bar{\lambda}_2 T\right)\right)} + \left(1 \vee \exp\left(-\bar{\lambda}_2 T\right)\right) \right) b \\
&\le \left(\tfrac{1-\exp\left(-\bar{\lambda}_2 T\right)}{\bar{\lambda}_2} + \tfrac{1 \vee \exp\left(-\bar{\lambda}_2 T\right)}{(1-\alpha-k_6 C_6)\left(1 \wedge \exp\left(-\bar{\lambda}_2 T\right)\right)} + \left(1 \vee \exp\left(-\bar{\lambda}_2 T\right)\right) \right) \\
&\quad \times \left[k_{14}^2 A + \left(k_{10}^2 + k_4 C_4\right) a + \left(k_{12}^2 + k_5 C_5\right) b\right] \\
&\le \left(\tfrac{1-\exp\left(-\bar{\lambda}_2 T\right)}{\bar{\lambda}_2} + \tfrac{1 \vee \exp\left(-\bar{\lambda}_2 T\right)}{(1-\alpha-k_6 C_6)\left(1 \wedge \exp\left(-\bar{\lambda}_2 T\right)\right)} + \left(1 \vee \exp\left(-\bar{\lambda}_2 T\right)\right) \right) \\
&\quad \times \left(k_{13}^2 \bar{B} + \left(k_8^2 + k_1 C_1\right) \bar{c} + \left(k_9^2 + k_3 C_3\right) \bar{d}\right) \\
&\times \left(k_{14}^2 \left(1 \vee \exp\left(-\bar{\lambda}_1 T\right)\right) + \left(k_{10}^2 + k_4 C_4\right) \tfrac{1-\exp\left(-\bar{\lambda}_1 T\right)}{\bar{\lambda}_1} + \left(k_{12}^2 + k_5 C_5\right) \left(1 \vee \exp\left(-\bar{\lambda}_1 T\right)\right) \right).
\end{aligned}
$$

第一个结论得证.

注意到

$$
\begin{aligned}
\bar{\lambda}_1 &\doteq \lambda - 2\lambda_1 - k_1 C_1^{-1} - k_2 C_2^{-1} - k_3 C_3^{-1} - k_7^2, \\
\bar{\lambda}_2 &\doteq -(\lambda + 2\lambda_2 + k_4 C_4^{-1} + k_5 C_5^{-1} + k_6 C_6^{-1} + k_{11}^2).
\end{aligned}
$$

如果

$$2\lambda_1 + 2\lambda_2 < -\left(k_7^2 + k_{11}^2\right),$$

我们可以取

$$\lambda \in R, \tilde{C}_i \in \tfrac{C_i}{k_i}, i = 1, 2, 3, 4, 5, 6,$$

满足

$$\bar{\lambda}_1 > 0, \bar{\lambda}_2 > 0, 1 - \alpha - k_2^2 \tilde{C}_2 > 0, 1 - \alpha - k_6^2 \tilde{C}_6 > 0.$$

则，我们有

$$
\begin{aligned}
&\bar{c} \le \tfrac{1}{\bar{\lambda}_2} \left[k_{14}^2 A + \left(k_{10}^2 + k_4 C_4\right) a + \left(k_{12}^2 + k_5 C_5\right) b\right], \\
&\bar{B} \le k_{14}^2 A + \left(k_{10}^2 + k_4 C_4\right) a + \left(k_{12}^2 + k_5 C_5\right) b,
\end{aligned}
$$

$$\bar{d} \leq \frac{k_{14}^2 A + [(k_{10}^2 + k_4 C_4)a + \left(k_{12}^2 + k_5 C_5\right) b]}{(1 - \alpha - k_6^2 \tilde{C}_6)}.$$

从而

$$\begin{aligned}
&\bar{c} + \bar{d} + \bar{B} \\
&\leq \left(\frac{1}{\bar{\lambda}_2} + \frac{1}{(1-\alpha-k_6^2\tilde{C}_6)} + 1\right) \times \left(k_{13}^2 B + (k_8^2 + k_1^2\tilde{C}_1)c + \left(k_9^2 + k_3^2\tilde{C}_3\right) d\right) \\
&\times \left(k_{14}^2 + \frac{\left(k_{10}^2 + k_4^2\tilde{C}_4\right)}{\bar{\lambda}_1} + \left(k_{12}^2 + k_5^2\tilde{C}_5\right)\right),
\end{aligned}$$

其中

$$\bar{\lambda}_1 = \lambda - 2\lambda_1 - \tilde{C}_1^{-1} - \tilde{C}_2^{-1} - \tilde{C}_3^{-1} - k_7^2,$$

$$\bar{\lambda}_2 = -(\lambda + 2\lambda_2 + \tilde{C}_4^{-1} + \tilde{C}_5^{-1} + \tilde{C}_6^{-1} + k_{11}^2).$$

第二个结论得证.

参考文献

[A] Antonelli F., *Backward-forward stochastic differential equations,* Ann. Appl. Probab. 3 (1993) 777–793.

[B] Bismut J.-M., *Conjugate convex functions in optimal stochastic control,* J. Math. Anal. Appl. 44 (1973) 384-404.

[EPQ] El Karoui N., Peng S., Quenez M.-C., *Backward Stochastic Differential Equations in Finance,* Math. Finance. 7 (1997) 1–71.

[F] Feyel D., Sur la méthode de Picard (EDO et EDS), in Séminaire de Probabilités, XXI, Lecture Notes in Mathematics 1247, pp. 515-519.

[HP] Hu Y., Peng S., *Solution of Forward-Backward Stochastic Differential Equations,* Probab. Theory Relat. Fields. 103 (1995) 273–283.

[MPY] Ma J., Protter P., Yong J., *Solving forward-backward stochastic differential equations explicitly—a four step scheme,* Probab. Theory Relat. Fields. 98 (1994) 339–359.

[MY] Ma J., Yong J., *Forward-backward stochastic differential equations and their applications,* Lecture Notes in Math., vol. 1702, Springer-Verlag, Berlin, 1999.

[PP1] Pardoux E., Peng S., *Adapted solution of a backward stochastic differential equation,* Systems Control Letters. 14 (1990) 55–61.

[PP2] Pardoux E., Peng S., *Backward Doubly Stochastic Differential Equations and Systems of Quasilinear Parabolic SPDE's,* Probab. Theory Relat. Fields. 98 (1994) 209–227.

[P1] Peng S., *Probabilistic Interpretation for Systems of Quasilinear Parabolic Partial Differential Equations.* Stochastics. 37 (1991) 61-74.

[P2] Peng S., *Problem of eigenvalues of stochastic Hamiltonian systems with boundary conditions,* Stoch. Proc. Appl. 88 (2000) 259-290.

[PS1] Peng S., Shi Y., *Infinite Horizon Forward-Backward Stochastic Differential Equations,* Stoch. Proc. Appl. 85 (2000) 75-92.

[PS2] Peng S., Shi Y., *A Type of Time-Symmetric Forward-Backward Stochastic Differential Equations,* C. R. Acad. Sci. Paris, Ser. I, 336(9): 773-778, 2003.

[PT] Pardoux E., Tang S., *Forward-backward stochastic differential equations and quasilinear parabolic PDEs*Probab. Theory Relat. Fields. 114 (1999) 123–150.

[PW] Peng S., Wu Z., *Fully Coupled Forward-Backward Stochastic Differential Equations and Applications to Optimal Control,* SIAM J. Control Optim. 37 (1999) 825-843.

[S] Shi Y., *Singularly Perturbed Boundary Value Problems,* ACTA Math. Appl. Sinica. 15(1999) 409-417.

[W] 吴臻, *正倒向随机微分方程理论及应用*, 山东大学博士论文, 1997.

[Y] Yong J., *Finding Adapted Solutions of Forward-Backward Stochastic Differential Equations—Method of Continuation*, Probab. Theory Relat. Fields. 107 (1997) 537-572.

致谢

谨在此衷心感谢导师石玉峰副教授。三年来，无论在学习上还是在生活中，我始终得到了导师的悉心指导和热情关怀。导师年轻有为，勤学不辍，在科研和工作上为我树立了榜样。我所取得的进步和成果离不开导师的辛勤培育。本文的开题、构思和撰写离不开导师的督促和悉心指导。在今后的工作学习中，导师对我的教诲和导师的敬业精神将永远激励着我。

衷心感谢彭实戈教授，他学识广博深邃，具有敏锐的科学直觉和独到的数学见解，治学态度极其严谨，使我体会到一种真正的科学精神。彭老师对于身边的学生，总是能够提供无私而热情的帮助，使我体会到一种人格的魅力。尤其感谢彭老师在本文撰写过程中给予的指导和帮助。

衷心感谢陈增敬教授、吴臻教授、赵卫东教授、林路教授、嵇少林副教授、刘杨老师以及史敬涛老师，感谢他们在学习和生活中给予我的种种帮助。

衷心感谢江龙博士、谷艳玲博士、张慧博士、冯春蓉博士、魏法金博士、杨维强博士以及我的同学吕峰、周俊华、王光臣、吴霜，在与他们朝夕相处中，我感受到了集体的温暖和力量。

衷心感谢郝鲁民老师、数学院孟玉梅老师、金融基地史春梅、孔云，她们在日常学习和生活中，给我很多的帮助，在此表达我由衷的感谢。

特别感谢我的父母，他们的爱护和期待是我人生道路上永恒不竭的动力，他们对我无私的爱，将使我从容面对一切人生的挑战。

感谢我的女友在生活上对我的帮助，她对我的鼓励和支持使我能顺利毕业。

正倒向重随机微分方程